# MATHEMATICS TEACHING

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TEACHING AIDS IN
MATHEMATICS

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# MATHEMATICS TEACHING

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## CONTENTS


Editorial .. .. .. .. .. .. .. .. .. .. 3

Recent Developments in School Mathematics. *Douglas H. Crawford* .. .. 7

Free Materials Suitable for Use in Teaching Mathematics .. .. .. 22

Geometry Board and Elastic Bands. *D. T. Moore* .. .. .. .. 25

A Demonstration of Pythagoras' Theorem .. .. .. .. .. 30

The O.E.E.C. Reports .. .. .. .. .. .. .. .. 35

Linear Programming in Schools. *I. C. N. Bell* .. .. .. .. .. 37

Experiment in Leicestershire Schools: the Dienes' Material. *David Wheeler* .. 45

Squares or Seconds? *C. J. Goold* .. .. .. .. .. .. .. 58

Secondary Modern School Mathematics – VII
G.C.E. and External Examinations. *Claude Birtwistle* .. .. .. 61

Notes and Notions .. .. .. .. .. .. .. .. .. 64

Apparatus Review .. .. .. .. .. .. .. .. .. 65

Letters to the Editor .. .. .. .. .. .. .. .. 66

Book Reviews .. .. .. .. .. .. .. .. .. 69

A.T.A.M. Diary and Announcements .. .. .. .. .. .. 76


No. 17 Winter, 1961 3/6

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# EDITORIAL

## RESEARCH IN EDUCATION

It is no longer necessary to argue that there is a crisis in the teaching of mathematics; the fact is now accepted. But opinions are still expressed as to how this has come about and what steps should be taken to meet the situation. Probably no single factor has contributed to the present position: there has been an increased demand for mathematicians, our technological society calls for a more thorough understanding of the subject by a larger number of people, the mathematics we teach has altered little in a society of rapid scientific and technical change, and the quality of teaching has been inadequate to meet the demands upon it; these are a few of the factors. But understanding how the situation has arisen merely shows us the problems to be reckoned with in overcoming that situation; it does not, of itself, give us a solution. It may reveal where lies the fault, but not what it is or how it can be rectified.

Nor is it sufficient to express opinions about what should be done; it is easy for one's opinions to be proved wrong in practice. And yet much of our British educational practice is based on opinion; the rest mainly on tradition. We have yet to see the application of scientific methods to the theory and practice of education in this country.

In the field of school mathematics much attention is being drawn to the fact that present-day syllabuses have changed little over the past two or three decades, and that there is little reflection of new mathematical techniques in what is taught at present. These opinions have been given support by the speakers at our Conference in London last Easter, by the recently published O.E.E.C. report (reviewed in this issue) and also by the recent Southampton Conference (the report is also reviewed in this issue).

But others sound a warning note, and quite rightly. It is not sufficient to have change simply for the sake of change. Are we certain we know where our ills lie? Will a change in *what* is taught necessarily solve our problems? Is it not perhaps *how* it is taught which is more important? And yet these questions themselves only hint at opinions. What do we know as of fact? Precious little it would seem. Yet no scientist or mathematician would work on data which could not be accepted as sound and unquestionable.

It is essential, therefore, that before we start to advocate reforms we should know something of the situation as it exists; before we start to suggest alterations we should experiment with our material and see what it is capable of doing; and before we put our suggestions into practice we should be certain of a high degree of success. All this calls for a very large experimental programme. Any research in education calls for a great deal of painstaking work on the collection of data, in many cases the result of carefully administered tests. This must be followed by careful diagnosis. Approaching the problem from the other end, if we are to introduce new material, its suitability should be tested both in relation to what is being done at the time it is taught and to what is to be studied later. Moreover, this must be done at different age-levels and different levels of ability. In other words, if we

are looking to reforms of this nature to bring about an improvement in mathematical attainment in this country, it must require an extensive research programme. After all, if one is hoping for such an improvement by this means it is clear it would have to be by radical and sweeping changes and not by the substitution of a few topics in the mathematics syllabus.

"There is a need for large-scale, systematic, scientific research in actual classroom situations where theories of subject matter, organisation and methods of instruction can be tested against controlled conditions over a long period of time. This research should aim to give knowledge on the quality and quantity of output in our educational system, as well as means for improving the quality of teaching. There is also need for fundamental research into problems of intellectual development and concept formation." So says the O.E.E.C. Report on *New Thinking in School Mathematics*. This theme is echoed in the conclusions drawn by D. H. Crawford in an important article appearing elsewhere in this issue. Shall we have to wait long to see the initiation of such a research programme in this country? That the time is ripe is evident; what we await is the inspiration which is to lead the investigations and the means to finance them.

Where can one look for the resources for such a research programme? Some in our Association are willing to experiment, but is this adequate? Many of them are already heavily committed with administrative work for the Association, and in any case it is doubtful if their field of experiment could be sufficiently wide. Nor, for that matter, are the finances of the Association adequate to stage a large-scale experiment. The National Foundation for Educational Research seems a likely place to expect this work to be done, but even here experiment is severely curtailed by lack of money; their Director recently stated that only 0.014 per cent of expenditure on education in this country went on research. Finally, one looks to the Universities and finds Education looked upon as the Cinderella of their Departments; most of the research is small-scale and a good deal is trivial in content. We have nothing to compare with the many extensive research programmes being undertaken in the U.S.A.

The time has come for a new attitude to education in this country. We have muddled through by rule of thumb for long enough. To-day, more than ever before, our national prosperity — perhaps even our national existence — depends on the quality of our education. Education is a craft, an art and a science, and the methods appropriate to each of these should be applied to its development.

In our own field a full-scale scientific investigation into existing practice of mathematics teaching is urgently needed alongside experiment with new ideas and full discussion between teachers of mathematics at all levels and pupil abilities. Only by extensive investigation of this nature can our mathematics teaching face up to the requirements of the future.

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# RECENT DEVELOPMENTS IN SCHOOL MATHEMATICS

## As seen from North America.

DOUGLAS H. CRAWFORD

*(This article is based on a talk given to the Canadian College of Teachers in the Autumn of* 1960. *Pressure on our space has prevented earlier publication. Our issue for May,* 1962 *will contain a further article in which the author will bring the material up to date).*

Today school mathematics stands at the threshold of a new, exciting, and challenging era in its history. Although mathematicians, teachers, and educators do not all agree on what exactly the future holds, yet there is general agreement that considerable overhaul is needed.

This article will commence with a brief sketch of how the present situation has come about. Next, an attempt will be made to describe, with illustrations, the type of revised mathematics which many people believe should be taught. This will be followed by a discussion of other developments judged equally important. Finally, a summing up of the overall picture, as the author sees it, will be given.

### 1. Causes of the Present Revolution

Many factors have combined to bring about the present dissatisfaction with school mathematics. Although the biggest upheaval over the issue is taking place in the U.S.A., yet the dissatisfaction is widespread and there is increasing concern in Britain and Europe as well as here in Canada.

Foremost among these factors must come the explosive development of mathematics itself. This development has been of two kinds: quantitative and qualitative. It has been quantitative in that much new subject matter has been created. Thus, branches of mathematics such as symbolic logic, topology, theory of games, and modern abstract algebra were little known a few decades ago. But just as important has been the fact that in the light of this growth, parts of the older mathematics have been reorganized, extended and transformed. An important example of this is algebra, which is now regarded as the study of mathematical structure. Newer and broader concepts have replaced those which used to be taught e.g. the concepts of sets and spaces, and emphasis on the properties possessed by a system defined by certain postulates is replacing the former emphasis on manipulative techniques.

At the same time, mathematics has been extending its applications into fields of activity in which, until recently, it had not been used. This is true both for traditional branches of mathematics (such as analysis and calculus, which are indispensable in engineering projects concerned with areas such as supersonic flight, space research, and automation) and for the newer more abstract branches of mathematics which are now finding increasing applications in the social sciences and industry. Thus economics makes great use of the theory of games and of linear programming, while industry, business and government itself are rapidly coming to depend on high-speed computing machines for purposes as diverse as assigning draftees to moderately suitable work categories, recording airline reservations, and conducting patent searches.

It is a distressing fact that both the school mathematics curriculum and the training of teachers of arithmetic and mathematics have failed to keep in step with these developments. Undoubtedly some of the blame must lie with university and research mathematicians themselves who in general have been too pre-occupied with their own specialities to take time to consider what should be taught in the schools and how teachers should be prepared to teach it. However, the demands of this technological era have revealed increasingly that a re-thinking of early mathematical training is essential if the supply of specialists trained in mathematics is to be adequate.

Finally, one can discern a number of sociological and philosophical factors which have also played their part.

One is the confusion between democracy and equalitarianism, particularly in the U.S.A. There has been in existence a notion that all children should be equal as far as the school curriculum is concerned. This has resulted in the view that "not to teach all children the same mathematics would be undemocratic." And so there has been an avoidance of streaming, and an adjustment of the learning rate to the abilities of the average or below average child.

A second sociological factor is simply this—that there is an abundance of challenging and remunerative employment outside teaching for the person who is competent in mathematics. Often this has been aggravated by a lack of a genuinely progressive and vital environment inside teaching.

Other related causes could be enumerated, but must be omitted. In summary, one might say this. The time is ripe for reform. The time for recriminations is past. Mathematicians, teachers, and interested educators have closed their ranks, and there exists as never before a wonderful opportunity to rebuild mathematical education on a sure foundation. This opportunity must be seized, and used wisely.

## II. The content and emphasis of the "New" School Mathematics in U.S.A.

*Introduction.*

Between 1820 and 1850 two events of major significance occurred in mathematics. The first freed the study of geometry from its ties with perception and 3 dimensional space, the second revealed that algebras, other than that which was essentially generalized arithmetic, could be constructed. Both of these events led to immense mathematical creativity, and to an increasing emphasis on the study of mathematical systems based on specified postulates or axioms. Side by side with this development came an increase in the rigor with which proofs were demonstrated. Later, towards the end of the 19th century, in investigations concerned with the idea of infinity, Cantor developed the theory of sets, and showed that this new concept provided a basic language in which arithmetical, algebraic and geometric ideas could be expressed.

Basically, programs of curricular reform are concerned that the material included should allow students to grasp these new ideas, especially the idea of mathematics as a dynamic growth in which the different patterns which result from different assumptions are of fascinating interest, and show unexpected applications to many

problems in the outside world. In practice, although some new subject matter, especially in Grades 10, 11 and 12, is proposed, much of the reform involves new precision of language, and a new approach to traditional content.

The drive for reform originated at college level, and has percolated first to high school level, and finally to grade school level, as it was gradually realized that to rebuild the upper storey while leaving the shaky foundation alone would only be a half-remedy.

*Illustrations of Suggested Changes.*

*A. The Elementary Level.*

Let us begin with the elementary school. The main strictures levelled at it have been that, due to the psychological and philosophical climate of the period 1920–1950, arithmetic has been fragmentized, and taught according to a criterion of social utility. Thus we read that in order to teach the addition of proper fractions some eighty different computational techniques should be taught, while content should be selected on the basis of how frequently it will be used by the child later as a citizen. This is not to say that there is no merit whatever in these points of view, but the fact remains that mathematics is the study of relationships and that both economy of time, and the way the human mind is constructed suggest that these principles should be subservient to teaching which emphasizes structure and concentrates on mathematical principles first.

One writer has coined a new phrase to describe some of the key concepts which are already beginning to influence school arithmetic. He calls them "the three s's" —"set, scale, and symbol." Let us use these as illustrative of possible developments, and add one more s–structure, so that we now have four s's instead of three r's.

1. The Idea of Set.

As mentioned already, this concept is basic in all mathematics, and is generally accepted as a key idea with which pupils should become familiar as soon as possible. However, it is wise to heed the caution that sets are a means and not an end, and that merely to use this language, and feel that you are teaching modern mathematics is to delude yourself.

Early in life a child learns to group things which have similar purposes such as crib, blankets, bottle and nap *or* spoon, glass, and dish. These are examples of *sets*. A set then is any collection of items or objects which belong together. To describe a set we may list its elements.

In the elementary school, the main purpose is to develop in students the ability to talk the language of sets. It should be noted that this does not imply a change in current methods of teaching, but adds variety and scope to problem exercises. Examples of its use might be the following. When teaching the basic number facts and relationships, we can ask the students to form the set consisting of all pairs of numbers whose sum is 5. These pairs are (0, 5), (1, 4), (2, 3), (3, 2), (4, 1). So we see that the set has only three members if order is neglected. As the child meets new numbers such as fractions, he learns that he is simply combining pairs of elements from the set of *integers* which he already knows. Thus the fraction 2/3 is the combination in the stated order of the integers 2 and 3 and could be written (2, 3) if we agree what this new symbol means.

2. The Idea of Symbol.

Another major concept which the child learns to use early in life is the *symbol*. Thus he soon comes to realize when his mother says "No" that his immediate whim will not be satisfied. Gradually he grasps the magic of words, and enters a world far removed from the physical world in which he actually lives. These oral symbols, or "language", as we call them, become his main means of communicating with other people. An important aspect of symbolism is that we use symbols to represent things. When I say "Canada" I am using that particular oral symbol to represent a vast volume of actual land, water, and mountains. The point as far as arithmetic is concerned is that children should be taught to recognize that the symbol '8' *merely stands for* the number 8. It is the *name* we give to the idea of eightness. Furthermore, the pupil ought to realize that numbers have many different names and that the study of the pattern and sense of these names is important.

Thus $16 = 7 + 9 = 8 + 8 = 9 + 7 = \frac{32}{2} = 2 \times 8$

A discussion of these equalities reveals many interesting things. Thus we see that $7+9 = 9 + 7$, or after numerous experiences that when we add two numbers, the order in which we do so is unimportant—as far as the result is concerned. This leads to an awareness of one of the underlying properties of our number system—the commutative law, a generalization which reduces memory work considerably. Again, if we know $8 + 8 = 16$, then we can deduce that $7 + 9 = 16$ from the fact that $(8 - 1) + (8 + 1) = 16$. These examples illustrate the close connection between symbols and the *structure* underlying our number system.

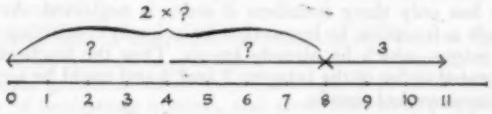
3. The Idea of Scale.

Addition and subtraction facts can be readily visualized on a number line or scale.

Thus the subtraction $5 - 2$ can be interpreted on the scale as directing the pupil to count 5 units forward, then two backward, ending with the number 3. The value of this lies in developing the connection between numbers and space as well as in helping the child to understand by means of visual aids. The number scale can also be used to give much experience in displaying the basic concepts of cardinal and ordinal numbers. Thus, in the diagram,

"Point to the third number after 1" is an ordinal or ranking use of number, whereas "Add 3 to 1" is a counting or cardinal use of it.

We can use both the number line and symbols to help introduce the child to algebra. Thus $2 \times ? + 3 = 11$ can be represented visually on the number line as follows.



4. The Idea of Structure.

We have already noted the intrusion of structure into the operation of adding two numbers. Now let us illustrate its value in two other connections. The fifst is concerned with division. Consider the problem 84362 ÷ 6. Let us set it down as follows:

a) How many 6's in 8 (ten thousands)? 1 ten thousand
Remove this 60,000. Symbolically

```
6)84362(                    10,000
  60000
  -----
  24362 is left
```

b) How many 6's in 24 thousands? 4 thousand
Remove 4,000 sixes or 24,000

```
6)24362(                     4,000
  24000
  -----
    362 is left
```

c) How many 6's in 3 hundred ? in 30 tens ? 5 tens or fifty?
Remove 300.

```
 6)  362(
     300                        50
     ---
      62 is left
```

d) How many 6's in 6 tens? 1 ten.
Remove 60.

```
 6)   62(                       10
      60
      --
       2
```

Add partial quotients: 10000 + 4000 + 50 + 10 = 14060.
Hence, 84362 ÷ 6 = 14060, with a remainder of 2.

This new method is based on the structure of arithmetic in several ways. First, it depends on a clear understanding of place value, second it reveals clearly that division is successive subtraction or shortened subtraction, and thirdly it utilizes what is known as the distributive principle. What we are really doing is splitting up the dividend as follows:

84362 = 60000 and 24000 and 360 and 2 and distributing the division over each of the parts. Although this method takes longer to learn initially, yet in terms of understanding it soon begins to pay "dividends"—to use the word out of its strictly mathematical context.

The second illustration can only be touched on briefly.

Consider these two problems.

1. May has six apples. Her mother gives her five more apples. How many has she now?

2. May has some apples. Her mother gives her five more apples. May now has 11 apples. How many apples did May have?

The point here is that the two situations are entirely different structurally. First the sense or structure of the problem must be appreciated, and only then should we attempt to represent it symbolically.

For 1, we have $6 + 5 = N$

For 2, $N + 5 = 11$

There must be more concern with the fundamental structure of the situation, if arithmetic is to be taught meaningfully and successfully.

B. *The Secondary Level*

It is at this level, particularly in Grades 9–12 that most activity is going on. Attention has been focused heavily on revised curricula embodying a certain amount of new content, but the claim is made that it is "the spirit rather than the letter" that counts. Most of the new material currently being explored emphasizes the structure of mathematics, its logic and its rigor, and the preciseness of its language.

Space precludes any detailed discussion of different suggested programs, but the names of three major groups should at least be mentioned. The first is the pioneer Illinois Group (or U.I.C.S.M.) which started in 1951, the second is the Commission on Mathematics of the College Entrance Examination Board (C.E.E.B.) which issued its report in 1959 after some four years of deliberation, and the third the School Mathematics Study Group (S.M.S.G.) which was convened in 1958, with the backing of teachers and university mathematicians. Historically, these three show how concern with High School Mathematics has grown to national proportions in the U.S.A., especially when one learns that besides producing textual materials, the S.M.S.G. group is writing monographs on such subjects as computing machines and cryptography for teachers and superior students, and is to study the use of films and T.V. in teaching and teacher training.

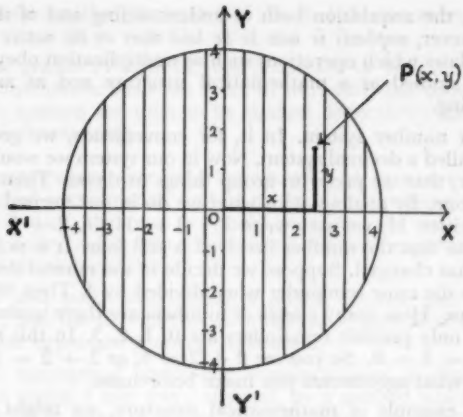
In order to illustrate the kind of new material proposed and its treatment, let us look briefly at some of the main suggestions of the C.E.E.B.

1. *The Concept of Set and Set Notation* which leads to the definition of a function as a set of ordered pairs. To try to give a glimpse of this, consider the following diagram (*top of next page*)

We would say that this is the graph of the function A, and write

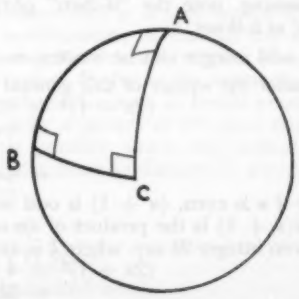
$$A = \{(x, y) \mid x^2 + y^2 < 16\}$$

where the meaning of the symbols is "A is the set of all ordered pairs (x, y) such that $x^2 + y^2$ is less than 16". The point P with 2 coordinates x and y in that *order* (hence the phrase ordered pairs) would be a typical point of the graph. It is fair to say that the study of equations, inequalities and graphs is clarified and deepened by this more precise language.




2. *The Revision of the Euclidean Geometry Course.*

A revision so that a combination of plane geometry, analytical geometry, and solid geometry replaces it. Included is a preliminary unit which discusses "if-then" or postulational reasoning in both mathematical and non-mathematical situations, together with the place of assumptions, definitions, and proof in the consideration of a deductive system. (Briefly, a deductive system consists of two parts. First, we have an "if" part, where we agree on certain assumptions, undefined terms, and definitions. Second, we use logical or deductive reasoning on the basis of our agreements to derive new knowledge. The results can be called the "then" part). This is followed by a short chain of theorems to show how the system is developed. The inclusion of coordinate geometry enables the student to enlist the aid of algebra as a geometrical tool, and lays the foundation for extension to $n$ dimensions later, and for future work on calculus. Finally, using spherical geometry and other pictorial material, the student can be led to appreciate that there are many other geometries, all of which have their own particular structure depending on the postulates and concepts on which they are based. A very simple example must suffice as illustration. Consider the spherical triangle ABC formed here by these three lines.




Clearly the sum of the three angles of this triangle is not 180°, but is 270°, showing that the geometry of the sphere differs from that of the plane.

3. *In algebra*, the acquisition both of understanding and of skills is recognized as essential. However, *emphasis is now to be laid more on the nature of number systems*, and on the basic laws which operations such as multiplication obey. In other words, algebra is to be studied as a mathematical structure and as an introduction to deductive reasoning.

Consider our number system. In it, for convenience, we group quantities in tens so that it is called a decimal system. Now in our system we would say $2 + 2 = 4$. Suppose, however, that we prefer to group things in threes. Then $2 + 2 = 3 + 1$ is one three and one. By analogy with what we do in our normal system, since one ten and one is written 11, we may write $3 + 1 = 11$ i.e. $2 + 2 = 11$ in our new system. Please note that the number involved is still four. It is only the notation or the name which has changed. Suppose we decide in the normal decimal system that $a = b$ if each has the same remainder when divided by 4. Then $39 = 23 = 3$ since $R = 3$ in each case. How many classes of numbers are there under this agreement? Only 4 since the only possible remainders are 0, 1, 2, 3. In this new system, what is $2 + 2$? $2 + 2 = 4 = 0$. So you see $2 + 2 = 4$, or $2 + 2 = 11$ or $2 + 2 = 0$, all depending on what agreements you make beforehand.

As a simple example of mathematical structure, we might consider the expression $x^2 + y^2$. Most people were taught in school that this could not be factorized, i.e. it could not be expressed as the product of two expressions, whereas $x^2 - y^2$ can be expressed as $(x + y)(x - y)$. The fact is that if we extend the scope of our discussion to include the set of complex numbers (numbers of the type $a + bi$, where $i^2 = 1$) it can. We would therefore modify our statement and say that in the set of real numbers, $x^2 + y^2$ has no factors. In the set of complex numbers, $x^2 + y^2 = (x + iy)(x - iy)$. It becomes very important to know just what the limits of our discussion are, and this idea is used considerably in dealing with equations and graphs.

A classic illustration of deductive reasoning in algebra is the proof that there are no natural (or counting) numbers a and b such that $a^2 = 2b^2$. However, to illustrate this aspect, consider the following problem.

"If an odd integer is greater than 1, then its square is 1 more than a multiple of 8".

In passing, note the "if-then" phrasing of the problem. We would proceed somewhat as follows.

Any odd integer can be written as $(2n + 1)$, where $n$ is itself an integer.

Consider the square of this general odd integer,

$$\begin{aligned}(2n + 1)^2 &= (2n + 1)(2n + 1)\\ &= 4n^2 + 4n + 1\\ &= 4(n^2 + n) + 1\\ &= 4n(n + 1) + 1\end{aligned}$$

Now if $n$ is even, $(n + 1)$ is odd and if $n$ is odd, $(n + 1)$ is even. Hence the product $n(n + 1)$ is the product of an odd integer times an even integer. But this is some even integer $2k$ say, where $k$ is an integer.

$$\begin{aligned}(2n + 1)^2 &= 4(2k) + 1\\ &= 8k + 1\end{aligned}$$

i.e. We have shown that the square of $(2n + 1)$ is one more than $8k$, which is a multiple of 8.

Later on, in grades 11 and 12, the properties of real numbers such as the associative and distributive laws are used to introduce the concepts of fields and groups, and such systems are then to be studied deductively using certain axioms as a basis for logical reasoning.

4. *In trigonometry*, vectors are to be introduced both as a very important concept in pure and applied mathematics, and as an excellent unifying idea in coordinate geometry. In this way, theorems in plane geometry, as well as many physical problems, can be solved using vector arithmetic. Further experience can, if desired, be gained with deductive systems.

Rectangular and polar descriptions of points, vectors, and complex numbers are intermingled, and the periodic or wave nature of the cosine and sine functions, which is their most important property, is stressed.

Finally, the concept of one-to-one correspondence is much used, and the circular functions cosine $x$ and sine $x$ derived, by a method which shows that these need not be defined in terms of angles, but are in fact sets of ordered pairs of real numbers.

The development is novel and interesting and allows of plenty of applications, and interrelationships with other branches of mathematics.

An illustrative example is omitted in view of space limitations.

These are, of course, not all the possibilities which have been suggested, which include a course in probability and statistical inference with a unit of descriptive statistics in Grade 9 (or even earlier in the S.M.S.G. program) followed by a theoretical development in Grade 12 using the theory of sets. Matrix algebra and symbolic logic used to test the validity of reasoning are other suggested items.

## III. Other Significant Developments

It would be wrong to give the impression that it is only in the U.S.A. that developments are taking place in school mathematics, or that all the attention is being directed to curricular revision. Among the research studies of importance, the following seem significant and encouraging. Probably the research which has affected thinking about the teaching of mathematics most in recent years is that of the Swiss psychologist Jean Piaget, who over a period of 40 years has patiently investigated how concepts such as those of number, space, and logical reasoning develop in children. In these variables, he has come to distinguish several stages of development, there being three main stages:

1. Pre-operational (2 - 7 years)
2. Concrete Operational (7 - 11 years)
3. Formal Operational (11 years and over).

In the first stage up to the age of about 7, the child's mental activity is dominated by partial sensory impressions. For example, he thinks that because water is poured from a broad based tumbler into a narrow based one so that the final level is higher, there is more water than at first.

In the second stage, he is capable of a limited form of reasoning, but it has to be tied to his own concrete experience.

Finally, in early adolescence, the ability to perform abstract operations becomes apparent. The child can set up an "if" situation and deduce the logical consequences.

Other investigators have confirmed these findings of Piaget in the main, although it has been shown that mental rather than chronological age is a better guide to the occurrence of the stages.

Independent researches by Hotyat and Johannot in Europe seem to indicate that mathematical concepts are not stable at least until the age of 13 or 14 years, and that not until well on in the secondary school can pupils argue meaningfully with procfs based upon general propositions.

Recent research by Dr. Robinson at the University of Alberta in Canada postulates that three types of mathematical ability exist with reference to the performance of a student in an axiomatic system. These are Level I—Comprehension, Level II—Problem solving, and Level III—Creativity. Studies at Level II seem to substantiate the major hypothesis that problem solving is limited by the available mental energy, and that upper bounds of performance are predictable for a majority of students after a short trial run in geometry.

In the U.K., Dienes has suggested that legitimate mathematical thought may be either analytic or synthetic. He believes that until now the whole educational approach has been based on the analytic mode of thinking and that learning may fail to take place if pupil and teacher belong to different types of thinking. He has put forward a theory of mathematics learning based on Piaget's findings. According to Piaget, learning takes place in organic cycles, and so Dienes has developed what he calls preliminary, structured, and practice games to fit in with each of the three stages of a particular cycle. To this *dynamic* principle of learning (i.e. the provision of such games) he has added three more—those of *constructivity*, *mathematical variability*, and *perceptual variability*. Thus, construction of experience of a concept should always precede analysis of it, since analytical thought does not appear in most children until the age of 12. The widest possible variation of the mathematical variables will enable a generalization to be formed from many experiences, while perceptual variability is essential so that genuine abstraction of the concept being formed is obtained, free from any particular perceptual structure.

Using this theory as a basis, together with the additional premise that each child learns at his own individual rate, he has developed concrete materials, notably the Multibase Arithmetic Blocks (M.A.B.) and the Algebraic Experience Materials (A.E.M.) together with sequences of learning and instruction cards designed to give students *individual experiential learning*. Results to date are highly encouraging, and Dienes is now planning work on mathematics learning from age 12 upwards. Further details may be found in the references.

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Gattegno, much of whose work has been done in Europe, especially in the United Kingdom, sees mathematics teaching in terms of situations and structures. He believes that we should create situations where children can discover relationships for themselves. His system of teaching arithmetic using coloured rods, originally developed by Cuisenaire in Belgium, is in line with many modern mathematical concepts, and has had great success in the early grades. He claims too that the rods can be used at secondary school level to develop topics such as permutations and arithmetic and geometric progressions. In Brentwood, New York, secondary school teachers are using the rods concurrently with new programs such as S.M.S.G. and reports indicate that they are very helpful with many topics in the programs, e.g. factoring and properties of the counting, integer and rational number systems.

The main value of the Cuisenaire rods seems to be for arithmetic and algebra although areas, volumes and extensions to $n$ dimensions can be illustrated too. Gattegno has also developed what he calls geoboards on which the knots of particular lattices are marked by nails round which elastic bands can be placed. In this way a vast number of geometrical situations can be produced, and geometrical structure investigated. Yet another way of creating situations and showing relationships is by means of films produced by Nicolet in Switzerland and Fletcher in England.

It is probably true to say, that mathematics in the elementary school has been number centred, and that concepts of space have not received any really systematic treatment. Current research at Stanford University in California claims to have revealed that students in the very early grades can grasp the concepts of points, and lines, and can make simple ruler and compass constructions. In addition, work on the development of spatial concepts by Piaget suggests that children may develop topological notions such as inside and outside even before they distinguish between Euclidean shapes such as squares and rectangles. Thus there may come considerable re-organization of elementary mathematics if these new investigations prove valid, for certainly in this space age, we need to develop important spatial concepts as early as possible.

*Developments in Canada*

These can be considered briefly under the following headings: Curriculum, teaching methods, teacher training and research.

In the realm of curriculum, at least three provinces have studies under way at the secondary level. In Ontario, the O.T.F. has set up a Mathematics Commission which held a workshop at Lakefield in the fall of 1959. As a result, some twenty teachers experimented during the winter of 1959/60 with a grade 9 syllabus containing the same sequences of topics but using set language and a treatment of inequalities together with greater emphasis on the structure of the number system. First results appeared to indicate that average students gain a surer understanding of algebra and analytic geometry in this way. Since then an experimental text has been produced.

In British Columbia, experimental work with four grade XI classes is proceeding to determine the feasibility of introducing the elements of set theory and symbolic logic at that level.

Most of the other provinces have begun to examine current U.S. publications. For example, New Brunswick has in operation a Mathematics Committee which has

discussed the necessity of revising the mathematics program from grades seven to twelve "so that students may develop an understanding of the spirit, method and content of contemporary mathematics." The committee has made a number of recommendations, and outlined both an experimental and an interim secondary mathematics program.

Alberta has decided to carry out a re-structuring of the entire program from grade 1 onwards. A three-year study of the elementary school program in arithmetic began in the fall of 1959 in order "to be fairly certain that the foundation program . . . is as good as we can make it before seriously considering a revision of the secondary school program in mathematics." Some experimental work is now being done at the secondary level.

To prepare teachers to teach contemporary mathematics with understanding, summer institutes and extension courses have been offered in a number of universities.

The major new teaching method which is being tried in provinces such as B.C., Saskatchewan and Manitoba is the Cuisenaire method of teaching Arithmetic (already mentioned). So far almost all the work has been confined to grades 1, 2 and 3, and the results are not sufficiently conclusive to justify categorial statements, but teachers using the rods are very enthusiastic, and there is evidence that children gain outstanding facility in the complex manipulation of whole numbers and fractions without sustaining a loss in basic numberwork achievement or in problem-solving ability. The author has experimented with the rods at grade seven level with a class of below average ability and is convinced of their value, although they still need to be integrated into a properly designed curriculum.

In the field of research, the study of Robinson has already been mentioned. Cuisenaire rod studies also come into this category. Another investigation of note is that of Sankey who attempted to compare the critical thinking ability of students under different conditions. In the next few years, more fundamental studies in concept formation, means of achieving insight, and on how motivation takes place are badly needed.

Finally, in April 1960 a seminar convened by C.T.F.* was held in Ottawa on New Thinking in School Mathematics. It was attended by some 50 delegates from across Canada representing teachers, provincial departments of education, university departments of mathematics, teacher training institutions and professional bodies like the Canadian Mathematical Congress. An encouraging outcome of the discussion was the resolution to establish a Federal Council of Mathematics Teachers.

*Further Trends*

In the writer's opinion, the most encouraging trends of the last year in the U.S.A. are two in number.

1. *An increasing recognition that learning depends on individual experience and discovery.* The work of Dienes begun in Leicester, England, has been further developed by him working with Professor Bruner at Harvard, and is attracting considerable attention. An attempt to apply similar psychological principles with students at Grade 9 level (Mathematics Individual Learning Experiment—M.I.L.E. for short) is being made

* C.T.F.=The Canadian Teachers' Federation.

using all kinds of supplementary perceptual materials, and responsibility for carrying out his program studies is largely that of the student. Apparently, encouraging results are being obtained. Emphasis on individual discovery is also to be found strongly in the experimentation of the Syracuse University Madison Project and others, and in the rapidly-developing field of "programmed learning" using so-called teaching machines.

2. *A growing attention to the basic problem of teacher training.* This is evidenced by Reports such as that of the Mathematics Association of America which, through a panel on Teacher Training, has prepared a set of recommendations of minimum standards for the training of teachers on all levels. The panel has classified Mathematics teachers into 5 levels, roughly as follows: elementary, Junior High School, Senior High School, Junior college, and college teachers. The report recommends for level 4, for example, a Master's Degree with at least two-thirds of the courses being in Mathematics. In terms of British education, such a teacher would be approximately equivalent to a capable sixth form master.

The National Science Foundation is also extending its assistance, for example, to support Summer Institutes planned over 3 year periods, and is involved in a program soon to be initiated at Webster College, Missouri to train elementary teachers of Mathematics.

The general implications for mathematics teaching of these developments and researches would seem to be the following:

1. The concept of readiness is very important, i.e. the child may not be mentally ready for certain stages of the mathematics curriculum. Thus readiness tests and curriculum sequences designed in accordance with knowledge of stages of maturation seem indicated. This is not to say, however, that maturation cannot be somewhat accelerated by supplying appropriate experiences, for Churchill in the United Kingdom has shown that this can be done with young children.

2. It seems clear from the researches of Piaget, Hotyat, and Robinson that the ability of a majority of children to do logical reasoning of the type required by present secondary curricula, far less by proposed secondary curricula, is called in question.

3. If the theories and teaching methods developed by Dienes are successful, far reaching changes in our thinking about mathematical education are on their way.

## IV. **Conclusion**

The present situation then in school mathematics could best be described as fluid and exciting. There is intense activity in the U.S.A.—mainly, though not all, connected with curricular problems, a notable exception being the work on evaluation and testing at the Minnesota National Laboratory for the Improvement of Secondary Mathematics being carried out for the S.M.S.G. Also some valuable experimental work with pupils of lower ability is being developed.

A cautionary word should be added to the effect that not all are wholeheartedly in agreement with the proposed curricular reforms. The main criticisms fall under two headings:

1. It is not the curriculum, but poor teaching which is at fault. This can be remedied by presenting the intuitive meaning of each mathematical idea or pro-

cedure in the setting which led to its creation, and by devising examples of applications to physical problems.

2. Students are not mature enough to study the structure of axiomatic systems. They should be doing mathematics and enjoying it. This criticism is supported by the research quoted.

The author's view is similar to that expressed in the following words by W. W. Sawyer:

"If we simply introduce *new knowledge* into our syllabus, that too will be out of date by 1980. Our basic task is *to prepare our students for the unknown*, to impart initiative and self-reliance.

This seems to me decisive. I welcome the use of modern topics if they encourage discovery and independent thought. I welcome the use of ancient topics if they encourage discovery and independent thought. This should be the first consideration."

The author believes that some curricular revision at both levels is called for. But there are many other factors which need to be considered. Chief among them naturally come the *learner* and the *teacher*. We must fit the curriculum to the learner, and not vice versa. Likewise the best and most up-to-date curriculum in the world will be useless if the teacher cannot teach it adequately.

Another major problem is how many programs in mathematics should be set up, and at what level.

In conclusion, the two key questions would seem to be

1. a knowledge from research of what the student can learn at a given stage of mental maturity, and

2. teachers who are competent both mathematically and pedagogically.

Given these two conditions, we can surely design more relevant and satisfactory curricula and so improve school mathematics beyond recognition.

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Boards can be purchased ready made (see our advertisement pages from time to time), but many teachers prefer to save money by making their own sets. The actual size of the board is not a matter of great importance; 9 to 12 inches square will be found most satisfactory in practice being neither so small that manipulation of the bands is unduly cramped nor too large for comfortable handling. They should not be less than 5/16 inches thick if of ply-wood and thicker if of soft-wood. It is sometimes possible to obtain a supply of 'offcuts' gratis or nearly so. If no woodwork room is available the local woodyard will usually supply boards cut to size at a reasonable price. Knocking the nails in is a time consuming, rather tedious business, but within the capacity of top juniors. $\frac{5}{8}$" brass escutcheon pins can be obtained from most ironmongers. A supply of rubber bands 3" long and preferably coloured is the only other expense, and the only recurring expense. It might be possible to interest a Parents' committee in making a set of boards for the Primary School.

There are two basic patterns for the boards, those having nails arranged in a square lattice and those having nails placed round the circumference of a circle. The base for the square lattice is marked with lightly sawn lines dividing the board into a set of identical squares with a large headed brass nail in the centre of each square. (Figure I). These marked lines are not essential but in use will be found to be a valuable aid.

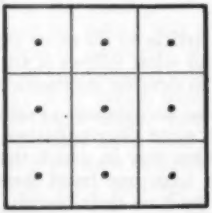


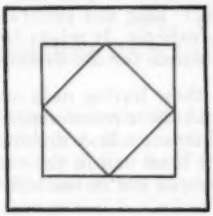
Fig. I.

One board per child is essential so that every pupil may obtain first-hand experience and have the opportunity to make discoveries for himself. For young children a square lattice of 9 (3 x 3) pins is the best with which to start. This provides enough opportunities to hold the children's interest for a considerable time and gives them a variety of experience without presenting situations which are

too difficult for them to handle; at the same time it provides a field within which it is possible to exhaust a given situation in a reasonable time.

The initial introduction should be a period of free play with a board and a few bands. If possible the boards should be left available for the children to take up in any spare moment. They will quickly realise that looping a band over the nails produces a shape which can be changed rapidly by simple manipulation. They will experiment with a variety of polygons, building patterns of all kinds. In this early experience the teacher should not interfere by imposing his own thought patterns or directing experiment along any particular lines, but careful questioning later on will show that from the start the child begins to recognise many basic structures on which mathematical concepts can be built. Ideas of congruency are an example: the right-angled triangle formed round 3 pins on a corner of the board can be copied on every other corner and produces a symmetrical pattern having a square (or diamond) in the middle. The frequency with which this centre square appears in pattern making often leads children to examine it carefully and to appreciate that it is indeed a square despite its orientation. (Figs. II and III). At this stage, of course, these perceptions will not be verbalized but it is clear that ideas of congruency, symmetry and similarity are developing rapidly, although no names will be given till a later stage.

Fig. II.



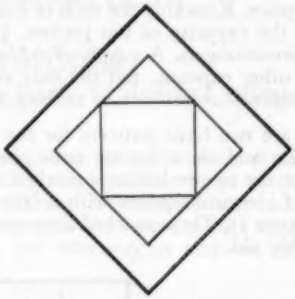


Fig. III.

It is not possible in a single article to do more than indicate some of the uses to which the board can be put and what follows is an attempt to show a few ways in which the board can be used to develop geometrical ideas.

Once the children have become accustomed to using the board and considering the resulting mixture of lines with some discrimination, the teacher can step in and start organising this experience. One way in which this start might be made is for the teacher to take a board and with one band form an asymmetric shape. He holds it up and asks the class to copy it on their boards. The class then has produced a set of congruent figures. The teacher turns his board round and asks the children to do the same while watching how the figure changes; they may also place the board flat on the desk and walk round it. As this goes on and the teacher is walking around the class with his board he is able to illustrate the two basic ideas of rotation and translation. The class could then perhaps consider the symmetry of the board, e.g. the way in which some shapes can be divided into two identical parts by a

diagonal or by the line joining the mid-points of opposite sides, or the significance of the centre pin; how a shape based on one corner can be repeated all round the board by simple rotation.

At a much later stage the board can be used in the introduction of co-ordinate geometry and the use of a co-ordinate system is of considerable advantage in many ways. Apart from its geometrical use, it helps as an early introduction to graphical work. For convenience and the purpose of this article the notation used will take the origin in the bottom left hand corner.

Consider various plane figures. Ask the children to place a band round any three nails. We may allow an entirely free choice but if we want to keep the discussion on formal lines it is better for the class to follow the teacher's pattern, say, (0,0) (0,1) (1,2). We give the figure its conventional name, triangle (tri-lateral would be better). Take one side and hook it over a fourth nail (1,1), giving a parallelogram, or taking it to (1,0) we find a trapezium. The trapezium can be deformed into a pentagon (2,1) and this into a hexagon (2,2), these being irregular. Removing two vertices from the hexagon produces in one case a kite, in another a square, in another a parallelogram or in the third case an irregular quadrilateral. The movement from one figure to another and back again is so slight and simple that the relationships between members of the set are easily and quickly built up so that when one figure is visible on the board the child is aware that all the others are potentially present. There is no need to hurry with giving names to the figures. When the teacher is satisfied that the children can recognize the obvious simple properties and are able to realize how these are affected by rotation of the board or by manipulation of the band then he will write the name on the blackboard and the children will copy it into their notebooks with a drawing, perhaps, to record what they have found on their boards. A rather dull looking heptagon is the only other polygon which can be found on the 9 pin board, but there are many ways in which the other polygons can be more closely examined. The family of triangles, of course, gives the greatest opportunities.

The right-angled triangle formed on the three nails in the bottom corner (0,0) (1,0) (0,1) can be reproduced in other positions on the board. We ask "How many?" and direct attention to the concept of congruency. Comparison of this triangle with the triangle formed by taking half the board, (0,0) (2,0) (0,2) gives an opportunity to introduce ideas of similarity. The triangle (0,0) (2,0) (0,1) and its rotations completes the set of right-angled triangles. The two scalene triangles (0,0) (0,1) (1,2) and (0,0) (0,1) (2,2) (again with rotations) considered with these help to clarify the concept of right angle. The set of isosceles triangles merits attention. The symmetry may already have been noted in two of the right-angled triangles already considered, this perhaps becomes most striking when we look at (0,0) (2,0) (1,2) and (0,0) (2,0) (1,1) and the discovery of the pair based on (0,1) (1,0) is tremendously exciting.

Other geometrical concepts may be explored on the 9 pin board, but as the children get older and more skilled in the manipulation of the boards and bands they will find more satisfaction in using a larger board. Certain situations which occur on the 9 pin board need expanding for the full interest of the general case to become evident. For example, on the 9 pin board we have only two parallelograms, (0,0) (1,0) (2,1) (1,1) and (0,0) (1,0) (2,2) (1,2), (with rotations and reflections);

only one of these has a pin inside making it easy to examine properties of the diagonals. The 16 pin board gives nine parallelograms and the 25 pin board gives 24 with at least six having a pin where the diagonals intersect. So that in order to generalize from experience and to verify (what can only be a suspicion on the 9 pin board) that the diagonals bisect, we need the 25 pins. Larger boards, say with 36 or 49 pins are, in general, too complicated for children to handle. I use a 49 pin board for demonstration purposes but do not expect children to use more than 25 pins.

Although the perceptual field is much larger, the 16 pin board introduces a restriction of its own in that it has no pin as a centre of symmetry and the exploration of the symmetry of the board makes an interesting lesson, or series of lessons. For example: "Make a shape using one band. Has it an axis of symmetry? Can you change one vertex so that the figure becomes symmetrical? Place a band from corner to corner of the board. Is this diagonal an axis of symmetry? Take a band and make a shape on one half of the board. Change boards with your neighbour and reproduce his pattern on the other half of the board." Or: "Make a figure on one corner of the board. Rotate the board and copy the figure on each corner in turn so as to preserve the symmetry of the board. Repeat with a different figure. See how many different ways you can do this." A similar approach to the 25 pin board will illustrate the differences and the much greater freedom of the larger board. "How many axes of symmetry has it?"

On the 16 pin board the children can find more polygons and more examples of each, while on the 25 pin board the range is even wider.

The board can be used for exploring the concept of area. We consider the idea of space coverage or amount of flat surface enclosed by a boundary and take as our unit of measurement the smallest square of the lattice. The area of various right-angled triangles is quickly seen to be half that of a square or rectangle whose area can be counted. (Fig. IV.) Parallelograms and trapezia can be treated as made up of squares and right-angled triangles and the actual counting becomes a relatively simple exercise, particularly if bands of a different colour are used to mark off the constituent parts. We then progress to shapes of greater complexity and encourage the children to choose any shape they like and find its area; they may set a problem for their neighbour again. Working on their own like this they quickly progress to very complex shapes and evolve highly complicated methods of arriving at the result by addition and subtraction of different figures. At a later

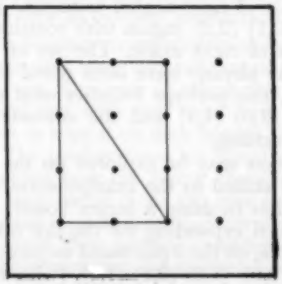


Fig. IV.

stage the rules or formulae for calculation can readily be generalized from this experience.

A simple exercise is:— "Make a right-angled triangle on the corner of the board" (0,0) (0,1) (1,0) "Double the length of the sides containing the right-angle. How many of the original triangles could you put in the new triangle? Now make the side three items as long. How many of the first triangle are included?" By the time "four times as long" is reached many children even at the age of 8/9 years will be ready to give the answer to the next question without troubling to make the figure on the board, and will generalise to a still greater length of sides. (*Euclid* VI, 19). Only an intuitive understanding, admitted, but a formal statement can come later as required and is much more acceptable, being based on previous experience.

Boards with a circular lattice are essential for the study of the regular polygons and unfortunately (from the economic point of view) each polygon requires a separate board which has only a limited field. For interest and excitement the regular hexagon is perhaps the best (See *Mathematics Teaching Pamphlet* No. 4). The regular octagon has almost as much to offer while the regular pentagon has the least, its main value being that it does produce a situation which is quickly exhausted in contrast to those arising on other boards. It is not reasonable to try to combine these on one board as a board with 120 pins on the circumference would need to be nearly a yard in radius and this is big even for a demonstration board.

The regular dodecagon is probably the best choice if only one set of boards with a circular lattice is possible. This enables the children to find an equilateral triangle, a hexagon, six equilateral triangles by dividing this hexagon, together with a variety of shapes giving opportunity for pattern and design. As an example of work which may be done with a hexagon board, the children are first asked to construct a hexagon and then insert a band to form a line dividing the hexagon. Usually they choose to join two opposite corners, thereby dividing the original figure into two trapezia. How many such trapezia could be made? (The rotations). Next use two bands, and in this case two triangles are obtained together with either a rectangle or a trapezium (Figs. V and VI). Three bands would give four triangles or three congruent rhombuses (Fig. VII). And so on to the final division of the hexagon into six equilateral triangles.

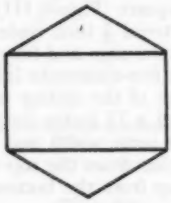


Fig. V.

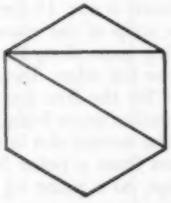


Fig. VI.

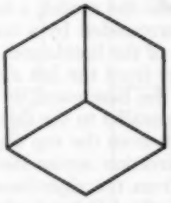


Fig. VII.

Throughout this work the teacher will almost certainly be supplementing the children's experience with diagrams on the blackboard and other more static visual

aids, and the children will from time to time record their discoveries or copy an interesting pattern of bands. Squared paper and coloured pencils are very valuable for this purpose of making a permanent record.

A fuller description of work that can be done with the boards will be found in *The Individual Geo-Board* by Dr. Gattegno (Publ. Cuisenaire Co.). But the greatest value of the boards lies in the discoveries that the children and the teachers make with them. Everyone who has experimented with rubber band geometry has some experience of value and the Editor would be glad to hear of any idea whether it can be expressed in two lines or would need two issues of the Bulletin.

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## A DEMONSTRATION OF PYTHAGORAS' THEOREM

Some while ago we received from Mr. D. S. Blacklock of Iver, Bucks. details of an apparatus which he had designed for demonstrating Pythagoras' Theorem. There is an inevitable delay in the publication of articles, and this enabled us to make a model based on Mr. Blacklock's diagrams. This has been used in teaching classes, where it proved effective, and it has been displayed at meetings of the A.T.A.M. to the interest of large numbers of people.

The model may be made in thick cardboard, hardboard or plywood. The last would probably be most effective as being both durable and smooth working in the sliding parts; our own model was made of hardboard. It consists of three parts: the baseboard and two moving parts which travel in slots cut in the baseboard.

The apparatus is shown in use in the diagrams. Figure I shows it in the closed position, when it can be seen that the whole square is made up of four congruent right-angled triangles and two squares which are clearly the squares on the two shorter sides of the triangles. The two movable pieces are now made to slide to new positions so that the model then appears as in Figure II. It can be seen that the whole square now consists of four congruent right-angled triangles (as before) together with one large square which is clearly the square on the hypotenuse of the right-angled triangles.

To make the model, a baseboard is cut, 15 inches square (Figure III) and this is partly surrounded by a narrow strip of the same material $\frac{1}{4}$ inch wide glued to the surface of the baseboard, leaving gaps along the bottom edge and for a distance of six inches from the left along the top edge. Two slots five-sixteenths inches wide are cut in the baseboard to allow for the free movement of the sliding parts. The first slot is parallel to the side, $3\frac{3}{8}$ inches away from it, and is $7\frac{1}{4}$ inches long, starting half an inch from the top edge. The second slot is of the same width and runs in a diagonal direction across the board from a point $3\frac{1}{4}$ inches from the top edge and $4\frac{1}{2}$ inches from the right-hand edge, to a point $1\frac{1}{4}$ inches from the bottom and $2\frac{1}{4}$ inches from the left-hand edge.

The two movable parts are in the form of trapezia, with one of the non-parallel sides perpendicular to the parallel ones. The smaller of these parts (Fig. IV) has parallel sides six and fourteen inches long and six inches apart; the larger (Fig. V) has parallel sides eight and fourteen inches long and eight inches apart.

Fig. I

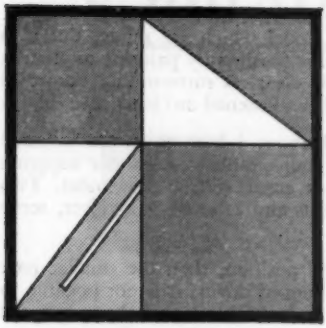


Fig. II

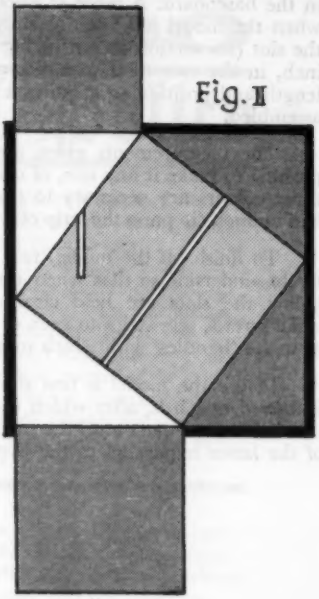


Fig. III

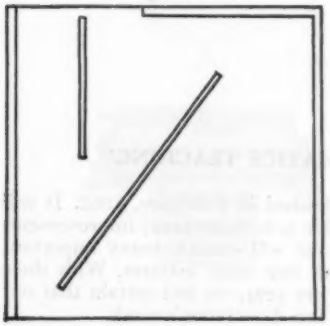


Fig. VI

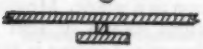


Fig. IV

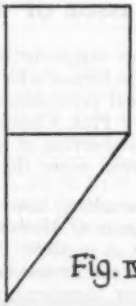
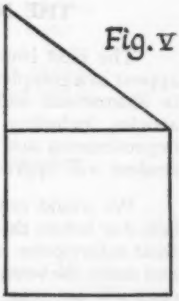


Fig. V



To the back of each of these is fixed a piece of wood slightly thicker than the baseboard and $\frac{1}{4}$ inch wide to act as a runner, and fixed so that it slides along the slots in the baseboard. A piece of wood about $\frac{3}{4}$ inches wide is fixed onto each of these when the model has been assembled, in order to prevent the runner rising out of the slot (see section shown in Fig. VI). The lengths of these runners is about one inch, in the case of the smaller one, and $2\frac{3}{4}$ inches, but the matter of the actual length and positioning requires a little thought and trial when the model is being assembled.

The measurements given are those of the model which we made, but it is possible to make it any size, of course. The model is finished by painting as shown; three colours are necessary to make the principle clear. A surrounding border of half an inch (in parts the strip of wood which has been fastened on) is painted black.

To finish off the model, two pieces of wood about $\frac{1}{8}$ inch thick are fastened to the underside so that when the model is laid on a table, the runner supports below the slots are held clear of the table thus ensuring free movement. Two small screws, screwed into each of the movable parts and allowed to project, serve as useful 'handles' with which to move the parts.

In use the model is first shown in the closed position, then the smaller part is moved upwards, after which the larger part is moved down; it is not possible to move the larger part until the small one has been moved. The slot for the movement of the latter is parallel to the hypotenuse of the triangle on the smaller part.

C.B.

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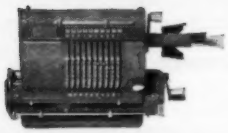
## THE NEXT ISSUE OF 'MATHEMATICS TEACHING'

The next issue of this magazine will be published in February, 1962. It will appear in a completely new format which will enable us to make many improvements in illustrations and general presentation. This issue will contain many important articles, including one by Prof. Choquet, and all our usual features. With these improvements and our production of four issues per year, we feel certain that our readers will appreciate even more the value of the Association's work.

We would remind members, however, that the subscription to the Association falls due before the next issue of *Mathematics Teaching*, and would ask them to renew their subscription as soon as possible. Prompt renewal reduces administrative costs, and assists the work of both the Treasurer and the Association as a whole. Membership costs £1 per year (student membership 10s.); subscriptions should be sent to the Hon. Treasurer, 122 North Road, Dartford, Kent. *Mathematics Teaching* is supplied free to members.

# ADDO

# Educational Policy



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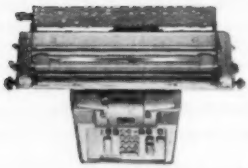

For some time we have felt that no one can legitimately criticise the standard of education of young people leaving schools and colleges unless and until industry and commerce provide education with (a) details of their requirements (b) reasons for these standards (c) know-how on how to achieve these, and (d) some practical help to enable equipment to be made available. This policy is an attempt to fill the gap and help out in the field in which ADDO are most active.

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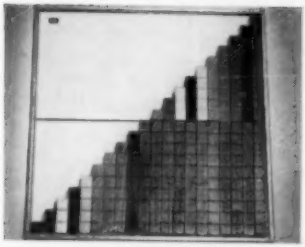

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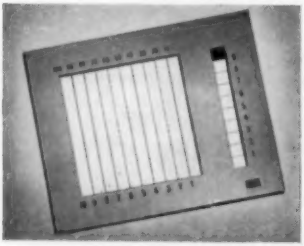

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## THE O.E.E.C. REPORTS

(a) "New Thinking in School Mathematics" (The Royaumont Report).
(b) "Synopses for Modern Secondary School Mathematics."
(c) "School Mathematics in O.E.E.C. Countries." (Summaries).

*Copies of (a) can be obtained (price 15/-) from any branch of Her Majesty's Stationery Office. Copies of (b) and (c) are obtainable free on application to O.E.E.C. Publications, 2 rue André-Pascal, Paris 16.*

These three documents, all published this year, are the written evidence of the recent activity of the Office for Scientific and Technical Personnel of O.E.E.C. The O.S.T.P. was created in 1958 to promote international action to increase the supply and improve the quality of scientists and engineers in O.E.E.C. countries (Canada, the United States and Yugoslavia are associate members). In the late autumn of 1959, a two-week Seminar was convened to meet at Royaumont, near Paris, to discuss "New Thinking in Mathematics." The report of this Seminar appears as Part 1 of volume (a). It had been hoped to have available for the members of the Seminar the results of a Survey in the form of answers to a questionnaire on mathematical education in the member countries. This was not, however, completed in time. Instead, each country was requested to provide a summary of its mathematical programme. These summaries form the text of volume (c) and a condensed account of the results of the Survey, completed during 1960, becomes Part 2 of volume (a).

The Summaries and the Survey, as concise reports of things as they are, have their moments of marginal interest, but the meat of the matter is contained in the Report proper and in the subsequent Synopses. The Report contains the full text of two important addresses: Dr. Marshall Stone (Chicago) on "Reform in School Mathematics," in which he persuasively puts the case for a change of existing syllabuses; and a challenging and controversial account, by Professor Jean Dieudonné (France), of how we should teach geometry when we have managed to get rid of Euclid. Abstracts only of the remaining addresses are given (among those reported are Willy Servais, Gustave Choquet, E. A. Maxwell and W. D. Wall) and one is sometimes tantalised by remarks too savagely summarised to be easily understood. But the general tenor of the Report is clear and remarkably uniform.

The Seminar agreed unanimously that it is necessary to modernise the teaching of mathematics. There are areas of disagreement over the ways in which this should be done, but, broadly speaking, the Report recommends an expansion of the place and importance of algebra in school mathematics together with the jettisoning of much of the trivial manipulation currently taught. Full use should be made of the language of mathematics as it has been developed during the last 50 years, leading to greater precision of mathematical expression and a gradual dissolution of the boundaries between the branches of the subject. Euclidean geometry should be considerably modified so that algebraic and vector methods can be introduced; trigonometry put in its place as a part of geometry and, later, analysis; probability and statistical inference introduced as respectable fields of instruction.

The Report makes it clear that it is not proposing a sudden and vicious change to a fully axiomatic treatment of mathematics at the 11+ Stage. It anticipates that the teaching will follow the usual pattern by beginning with intuitive ideas and gradually formalising them. For this reason these proposals need not be regarded as applying only to the most able children (although these are certainly the pupils most in mind throughout the Report).

The problems of implementing proposals for syllabus changes are considered and two specific recommendations made: (i) that those countries willing to work along parallel lines in the modernisation of syllabuses should receive financial support from the O.E.E.C. (this recommendation has apparently been approved); and (ii) that a group of experts should prepare a detailed outline of Secondary school mathematics indicating various modes of treatment for the benefit of those countries wishing to experiment with reformed syllabuses. Accordingly a group of sixteen mathematicians and teachers met for a month in the summer of 1960 and produced report (b).

These experts, although failing for lack of time to consider the whole of the syllabus for secondary school mathematics or survey the teaching at all ability levels, have produced in the Synopses a gold-mine of ideas and suggestions. They have considered the secondary stage in two parts, roughly ages 12-15 and 15-18, and produced detailed proposals for the teaching of some of the algebra, geometry, and probability and statistics, for these two cycles. The treatment of the first cycle is designed to be most easily modified for less able children (it specifically considers only the upper half of grammar streams), while the treatment of the second cycle is fully academic and intended as a preparation for subsequent university study. A whole range of difficulty is encountered by the reader from a relatively easy and familiar treatment of first cycle geometry to an outline of an axiomatic treatment of affine spaces for the best 18 year-old pupils. The book has not the unity of approach or style that makes for easy reading and there must be few teachers who can read it straight through. But the Synopses is essential for those teachers who have a nodding acquaintance with modern mathematics and want to find out more about its relevance to school teaching.

Those members of the Association who attended the Berridge House conference will find much in these volumes that echoes the main theme there. Other readers will also see in these reports many things that have been said or hinted at in other issues of *Mathematics Teaching* and will understand the enthusiasm (if still a little wary) that some of us feel for the general direction of these recommendations.

It will be interesting to see how the member countries of O.E.E.C. react to the proposals made in the Report. Nobody can doubt the enormous difficulties that will have to be met — not least in this country — before reforms on the lines proposed can be effected. If there is to be a revolution it seems likely to be gradual. This, though, is to be welcomed if it encourages us to use the interval in preparing ourselves for a major effort of reorientation. We need new textbooks, courses for teachers and teacher-trainers, and pilot experiments. If we work at these things for the next two or three years I doubt if we shall find we have wasted our time.

D.H.W.

# LINEAR PROGRAMMING IN SCHOOLS

I. C. N. BELL

Linear programming is a 'new' branch of mathematics which has grown with the development of computers capable of performing the tedious arithmetic associated with it, but the basic ideas are simple and there seems no reason why they should not be introduced at secondary school level.

A problem in linear programming is one in which it is desired to maximise or minimise a function of several variables which is subject to linear constraints. Three simple but typical problems are given below, and it is instructive to give these to the pupils without any hints and ask them to find solutions.

1. A factory can use its machines and men to make either Ball Pens or Radios or a combination of the two. In order to make a weekly profit of £100, if pens are made it is necessary to use 20 machines and 1 man, while for the same profit it requires 8 machines and 6 men if radios are made. The factory has available 200 machines and 24 men. How should it divide its resources so as to make the maximum profit per week?

2. How should the management of the factory above proceed if, with the same resources, it can also divert some men and machines to the production of tin openers which require 10 machines and 3 men to make a weekly profit of £100?

3. An oil company owns 17 tankers of which 3 are in port A, 6 in port B and 8 in port C. The tankers are to be sent to refineries P, Q, R and S, the numbers of ships required at the refineries being 2, 4, 5 and 6 respectively. The table shows the costs to send a tanker from any port to any refinery; e.g. to send one tanker from A to P costs 10 units.

| | P | Q | R | S |
|---|---|---|---|---|
| A | 10 | 16 | 12 | 15 |
| B | 12 | 14 | 13 | 13 |
| C | 16 | 16 | 12 | 14 |

What ships should be sent where to achieve the transportation in the most economical method?

A number of children may find the correct solution to No. 1 by assuming that all the men and machines are used and solving the simultaneous equations:

$$20x + 8y = 200 \quad \text{(Number of machines)}$$
$$x + 6y = 24 \quad \text{(Number of men)}$$

where $x$ = number of £100 units of profit made from pens
$y$ = number of £100 units of profit made from radios.

These equations give $x = 9$, $y = 2\frac{1}{2}$ and a profit ($£P$) of £1150, where $P = 100\,(x+y)$.

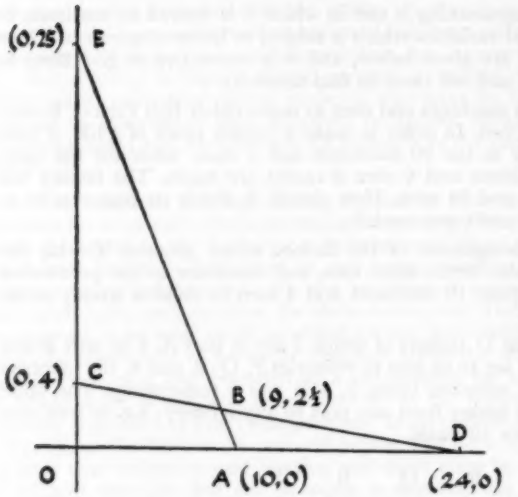
However, the question arises as to whether this is the maximum profit, for it may not be necessary to use all the men or machines. To find the maximum profit

we write the constraints (numbers of men and machines) as inequalities, so that in linear programming terms the problem is to maximise $P = 100\ (x+y)$ subject to the constraints

$$20x + 8y \leq 200 \quad \text{(i.e. } 5x+2y \leq 50\text{)}$$
$$x + 6y \leq 24$$

and

$$x \geq 0$$
$$y \geq 0$$




We now represent these constraints graphically. In the figure CD is the line $x+6y=24$ and AE is the line $5x+2y=50$. It will be seen that CD divides the plane into two halves and any point in the half-plane below CD satisfies $x + 6y < 24$, while for any point above CD, $x + 6y > 24$. Thus by virtue of the four constraints above we need consider only points inside or on the quadrilateral OABC since any points outside this will not satisfy one or more of these inequalities. Now the problem reduces to finding the point in OABC which maximises $P$ and so we consider the family of lines $x + y = \frac{P}{100}$. These are a set of parallel lines of gradient $-1$, and $P$ is a maximum when the distance from the origin is a maximum. This occurs at B $(9,2\frac{1}{2})$ when $P = 1150$.

The interest of this in school is not only that it shows that this is the maximum profit, but also in a discussion of other cases (Ref. 1). What happens when the function we wish to maximise is $P = 10x + 4y$? Are there impossible cases? Suppose the number of men given in the answer is not integral? What about question No. 2? We shall evidently need three variables here so we must use three-dimensional co-ordinate geometry. Then we find that with two constraints (other than $x$, $y$, $z$ positive) we reach a maximum by making pens and tin openers only. On the other hand with only one constraint (say the number of men is unlimited) we should make a maximum profit by concentrating on radios. Is this idea of *one constraint = one product* general? Suppose there were four variables what should we do then; use four dimensional geometry? Here the idea of an analytical solution, for example the Simplex method (Ref. 2) could be mentioned and perhaps examined in some detail, bringing in its suitability for computer use, since the arithmetic

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involved can become tedious. A problem solved in this way appears in an article by S. Vajda in the July 1959 issue of *Mathematics Teaching*, and I have re-solved No. 1 using this method in the appendix.

Although the transportation problem (No. 3) can be solved by the standard methods, it is of such importance that special methods are used, and these again are not difficult in essence (Ref. 2). This, too, is solved by one method in the appendix, and a number of interesting points arise — for example, no difference will be made to the routing if a constant is subtracted from each term in any row or column in the cost matrix.

It is important to emphasise the practical nature of this subject and to point out that in practice the matrices are very large and the arithmetic involved is long. A typical transportation matrix may be $150 \times 150$ or more and would take perhaps 6 weeks to solve by hand, though only 20 minutes on a computer. (If you don't happen to have a computer you can hire time on one for say £1 a minute). In an oil company there will be a staff of 35 - 40 persons working on linear programming problems such as transportation of fuels, storage, optimum blendings of fuels of differing octane ratings, etc.

It seems to me that here we have a subject which is of interest in the classroom because it is new and different, has important practical applications, extends our ideas of the interpretation of straight line graphs, raises a number of interesting questions and can be appreciated to some extent by pupils of almost any age in the secondary school.

## APPENDIX

*Solution of No. 1 using Simplex method:* This is rather clumsy for such a simple example, but can be applied to examples with any number of variables. Starting from our original inequalities we introduce two slack variables $a$ and $b$, both positive, so that the equations become

$$5x + 2y + a = 50$$
$$x + 6y + b = 24$$

where $a, b, x, y$ are all positive
and we wish to maximise $P = 100(x+y)$
*1st solution:* Let $a = 50, b = 24, x = y = 0$
Thus $P = 0$.

In this solution, $a$ and $b$ being non-zero are called the *basic* variables and $x$ and $y$ are non-basic. We express the basic variables and $P$ in terms of the non-basic variables:

$$a = 50 - 5x - 2y$$
$$b = 24 - x - 6y$$
$$P = 100(x+y)$$

Any increase in either $x$ or $y$ will increase $P$, so we increase $x$ as much as possible without making $a$ or $b$ negative; i.e. we increase $x$ to 10. We thus arrive at
*2nd solution:* $x = 10, a = 0, y = 0, b = 14$
Thus $P = 1000$

Our new basic variables are $x$ and $b$, and proceeding as before expressing them in terms of the other variables,

$$x = (50 - 2y - a)/5$$
$$b = (70 + a - 28y)/5$$
$$P = 20\ (50 + 3y - a)$$

Now any increase in $a$ will decrease $P$, but increasing $y$ will increase $P$, and we can increase $y$ to $2\frac{1}{2}$ before $b$ becomes negative. Thus we have

*3rd solution:* $y = 2\frac{1}{2}$, $b = 0$, $a = 0$, $x = 9$
and $P = 1150$

With $x$ and $y$ as basic variables, we have

$$x = (126 - 3a + b)/14$$
$$y = (70 + a - 5b)/28$$
$$P = 25\ (46 - 5a/7 - 3b/7)$$

Any increase in $a$ or $b$ will decrease $P$, and in order to alter $x$ or $y$ we must increase $a$ or $b$; that is, this solution maximises $P$.

The relation between this and the graphical solution is of interest. The first solution corresponds to the point 0 on the graph, the second solution to A and the third solution to B. If after the first solution we had increased $y$ instead of $x$ we should have reached B via C instead of A. Again the line OA is $x = 0$, OC is $y = 0$, AB is $a = 0$ and BC is $b = 0$.

*Solution of No. 3.* We find any feasible solution and then try to improve on it. A first solution is best found by the 'North-West Corner Rule'—starting from the top left-hand corner and filling in ships to agree with the numbers required at each port:

| | P 2 | Q 4 | R 5 | S 6 |
|---|---|---|---|---|
| A 3 | 2 | 1 | | |
| B 6 | | 3 | 3 | |
| C 8 | | | 2 | 6 |

This gives a cost
= 20+16+42+39+24+84
= 225 units

We may improve on this by using one or more of the unfilled routes and adjusting the other routes to agree. With a large matrix this hit-and-miss method is obviously unsatisfactory, so we proceed as follows:

Allocate a fictitious cost to each column and to each row so that the fictitious cost for any route is the sum of the row and column costs which intersect on that route, and so that fictitious costs for the routes already filled are equal to the actual costs.

Start with a f.c. of 0 for row 1

To agree with route A-P, the f.c. for column 1 is 10
" " " " A-Q, " " " " 2 is 16
" " " " B-Q, " " " row 2 is –2
" " " " B-R, " " " column 3 is 15
" " " " C-R, " " " row 3 is –3
" " " " C-S, " " " column 4 is 17

The table below is the same as the one above but with the addition of the f.c.'s for the rows and columns on the right and underneath respectively. Moreover, in brackets, for each of the unfilled routes is tabulated the fictitious cost minus the

true cost. For example, on the unfilled route A-R the f.c. is 0 + 15 = 15, while the true cost is 12; thus 15 − 12 = +3 is entered in the table.

| | P<br>2 | Q<br>4 | R<br>5 | S<br>6 | |
|---|---|---|---|---|---|
| A 3 | 2 | 1 | (+3) | (+2) | (0) |
| B 6 | (−4) | 3 | 3 | (+2) | (−2) |
| C 8 | (−9) | (−3) | 2 | 6 | (−3) |
| | (10) | (16) | (15) | (17) | |

It can be shown ("the proof is left to the reader") that if the f.c. is greater than the true cost for any route, then it will be advantageous to use that route. Thus here we shall use route A-R. The way to add ships to A-R is to take them from B-R, add to B-Q and take from A-Q. But the greatest number we can take from A-Q is 1, so we do this, leaving the following situation:

| | P<br>2 | Q<br>4 | R<br>5 | S<br>6 |
|---|---|---|---|---|
| A 3 | 2 | | 1 | |
| B 6 | | 4 | 2 | |
| C 8 | | | 2 | 6 |

The cost now is 222 units, and this is our second solution.

Applying fictitious costs again, as before, the next stage reached is

| | P<br>2 | Q<br>4 | R<br>5 | S<br>6 | |
|---|---|---|---|---|---|
| A 3 | 2 | (−3) | 1 | (−1) | (0) |
| B 6 | (−1) | 4 | 2 | (+2) | (1) |
| C 8 | (−6) | (−3) | 2 | 6 | (0) |
| | (10) | (13) | (12) | (14) | |

Now we use route B-S, taking 2 from B-R, adding 2 to C-R and taking 2 from C-S, giving

| | P<br>2 | Q<br>4 | R<br>5 | S<br>6 |
|---|---|---|---|---|
| A 3 | 2 | | 1 | |
| B 6 | | 4 | | 2 |
| C 8 | | | 4 | 4 |

This gives the cost of 218 units as our third solution.

If fictitious costs are again applied, it will be found that no alternative route will give any advantage, and it will also be seen that it is not possible to alter the numbers of ships on each of the present routes without using other routes as well, so 218 units is the minimum cost and occurs with the routing as shown above.

The above example has been chosen to avoid difficulties which sometimes arise — for example there may be several solutions giving the same minimum cost (what happens to our table then?), or the number of routes needed may not be the same as the number of constraints, but these are beyond the scope of this article and are dealt with in the books on this subject.

*References:*

1) Kemeny, Snell & Thompson: *Introduction to Finite Mathematics* (Prentice-Hall), 1956, Chapter VI.

2) S. Vajda: *An Introduction to Linear Programming and the Theory of Games* (Methuen), 1960.

---

## THIS TAKES THE CAKE

A boy went into the school tuck shop and purchased a quarter of a square cake which was on the counter; the piece was cut as a square from one corner of the cake leaving an 'L' shaped piece. When he had left the shop four more boys came in and asked how much the remainder of the cake cost. The shopkeeper said that he would give them the cake if they could cut the remaining portion into four parts equal in size and shape (cutting in layers is not allowed). How did the boys cut the cake?

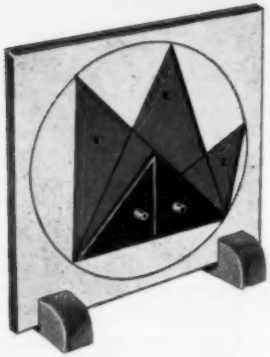
## Answer to 'Plain Dotty' in our last issue.

19801 dots. There are many ways of arriving at the solution. A simple one is derived from the observation that each of the arrays of dots in the sequence is the sum of two consecutive square arrays.

e.g. to obtain the fifth term combine $5^2$ with $4^2$ thus:

Therefore the 100th term will be the sum of $100^2$ and $99^2$.

# MATHEMATICAL MODELS AS VISUAL AIDS



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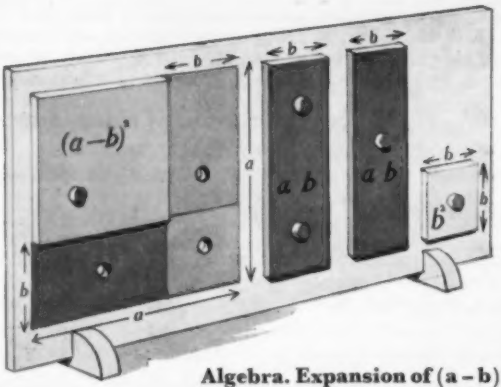


**Algebra. Expansion of $(a-b)^2$**
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# EXPERIMENT IN LEICESTERSHIRE SCHOOLS: THE DIENES' MATERIAL

DAVID WHEELER

[It is too early to provide anything like a final assessment of the methods in use in 60-odd Leicestershire Junior Schools and a few High Schools for teaching mathematics. This is an interim account for the benefit of those who have not seen the materials or heard about the methods. I have described a few of the stages in the method in some detail in order to give the 'feel' of the experiment, but for a fuller account the reader should see "Building Up Mathematics" by Z. P. Dienes (Hutchinson, 1960) and the teachers' manuals and pupils' work cards issued with the materials. The M.A.B. and A.E.M. materials are available from the National Foundation for Educational Research, 79 Wimpole Street, London W.1.

I have made comments as and when I thought necessary and the responsibility for these is my own, but I should like to acknowledge with gratitude the help of Mr. L. G. W. Sealey, Adviser for Junior Schools in the County of Leicestershire, and of those headteachers and teachers who have talked to me about the experiment and allowed me to observe their pupils at work.]

## Introduction

At this time, paralleling professional and public concern for the quality of mathematics teaching, the flow of new materials and new methods into schools has increased considerably in rate. In the last ten years, to take only the field of Primary School arithmetic, the market in this country has seen the arrival of structural material for teaching number work associated with the names of Cuisenaire ("Numbers in Colour"), Stern ("Structural Arithmetic"), Dienes ("Multibase Arithmetic Blocks"), Jones ("Avon" apparatus) and Shaw ("Structa" apparatus); new methods attached to these materials and to books by Bass and Dowty, Flavell, and so on; as well as a host of materials and suggestions of variable quality issuing from the major educational suppliers. In the midst of this flurry many Primary School teachers must wonder what, if anything, to do. It is understandable if with so many directions in which to turn many of them continue doing the same as before. The actual number of teachers actively and enthusiastically using any of these materials is still a small proportion of the whole[1] although there are signs that it is on the increase. Clearly, with so much on the market, it is hardly possible for every teacher to try out all these methods with equal objectivity and attempt to reach a firm conclusion. But it is very important for *some* teachers to put these materials to the test and eventually put their conclusions on record. It is perhaps unlikely that these records will give a precise pointer to one method as being the best of all possible methods, but the cumulative effect should be to show what the materials can do and what they cannot do.[2]

At this stage, it will become necessary to judge the relative value of the various abilities and disabilities of the materials. It may not matter if material "A" is not very good for teaching concept "X" if concept "X" is not found to be important. Material "B" may be admirably adapted to teaching concept "X" but that will not help it if it is decided that concept "X" has no place in the syllabus. In most

cases the decision isn't going to be as clear-cut as this and it will have to be reached by a mixture of mathematical, pedagogical and psychological principles. The matter is complicated, too, by the fact that a new material may put within reach concepts which were previously accepted as unteachable or inappropriate. So the principles used in judging will not remain static but will be influenced by the materials themselves—a new variable entering the situation and changing its character.

All this may make life for the teacher more difficult. If standards are changing how shall we judge? Well, we can only do as we always do—judge with the standards operative at the time of judgment: but with as much openmindedness as we can achieve and a willingness to accept that eventually we may be seen to be mistaken.

This is what I shall try to do here.

**The Experiment**

Rather more than 60 Junior Schools in Leicestershire are using a method for teaching mathematics devised by Dr. Z. P. Dienes, formerly Lecturer in Mathematics in the University of Leicester. This experiment has the support of the Director of Education and of the National Foundation for Educational Research. Some of the Junior Schools involved are also part of the Leicestershire High School Experiment and their pupils do not sit an 11+ examination. The method is gradually, but not universally, being introduced into the secondary schools (High Schools) as the pupils reach the age of transfer. The teaching scheme is not yet complete and the work cards are in process of revision at the present time.

No school uses this scheme as its only work in mathematics.

**The Theory**

Dienes distinguishes two main types of mathematical thinking: the *analytical* and the *constructive*. The former is, broadly speaking, a logical analysis of a situation whereas the latter is a method of building a new situation from prior experience. These two forms coexist in mature persons with a tendency for one or the other to dominate in a particular individual. The ability to pursue analytical thinking (a requirement of most traditional teaching) arrives later in life than the ability to think constructively and this leads Dienes to suggest that situations which are structured to permit the constructive approach to learning should be used in the early stages.

It is necessary to provide special situations which are structured towards a particular mathematical concept as the situations normally encountered in everyday life are not rich enough for more than a few mathematical concepts to be acquired through them. These special situations will encourage dynamic learning if they allow for three stages:

*Preliminary 'games'* in which the response is fairly undifferentiated
*Structured games* deliberately designed to lead to the formation of the concept
*Practice games* which utilise the concept and anchor it in experience.

The differences in ways of thinking also imply that the child should be given freedom to formulate his thinking in his own way and not have a view, which may be alien to his particular way of thinking, imposed upon him. This freedom demands the presentation of a structured task in as many perceptually different forms as possible.

Again, since mathematical concepts are those relationships which remain constant when certain aspects of a situation are changed, the composition of tasks must allow for the variability of all those factors which are independent of the relationship to be found.

**Classroom Implications**

The children work individually, or in pairs or small groups, from assignment cards. The teacher acts as a guide and counsellor, not as instructor. His job is to give help when it is needed (and equally to withhold it on occasion) and to make sure that the child is making progress at his own best rate.

The practice steps for each task are devised by the child. He may be given general instructions or a typical example and then asked to perform the task several times in different specific instances until he understands the task and can proceed to the next.

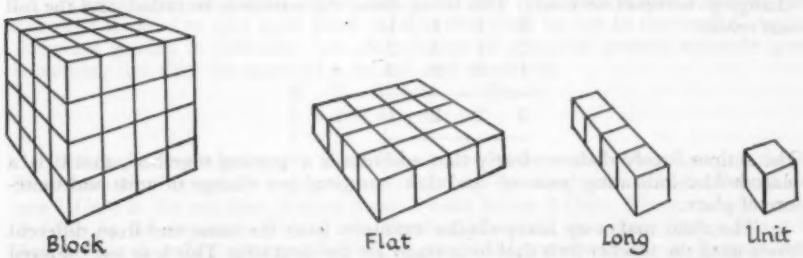
Throughout the scheme each concept is approached from a basis of experience with concrete material and a task to be performed with it. The forms of this material are varied as much as possible for each kind of task.

**The Scope of the Material**

The schemes of work can conveniently be considered in two sections, one mainly arithmetical and the other mainly algebraic. Most of the arithmetic in the scheme is covered in the first two years of the Junior School. The algebra is introduced in the last two years and continued into the High School. It is hoped to extend the scheme to cover all the work needed for GCE O level. (It is difficult at the moment to lay down the age levels at which it is best to attempt the various sections of the scheme; partly because the children, of design, work at their own rate; partly because the experiment has not been in operation long enough for more than one small group of children to have worked through the scheme from the beginning. The experiment began about three years ago with a small number of schools and has been extended to the present number of schools in stages since that time).

**The Blocks**

The Multibase Arithmetic Blocks (M.A.B.) are provided in several sets, each set having one of the bases 3, 4, 5, 6 or 10. In the Base-4 box there is a number of pieces of the following kinds:



The pieces are of wood, uncoloured, and grooved so that the structure of each piece in terms of small cubes can be seen. All the arithmetical tasks are performed with material from most of the boxes, chosen at random.

Following on unstructured play with the material, the child is gradually introduced to exercises in which he takes random samples of each kind of block from a particular box. These are put in piles in a row. *Blocks* at the left and *Units* at the right. He is instructed to change any pieces into large ones if he can and record his answer. If he originally took 2 *Blocks*, 5 *Flats*, 1 *Long*, 7 *Units* from the 4-box, he will change four of the *Units* into a *Long* and four of the *Flats* into a *Block* and will record his result as:

| B | F | L | U |
|---|---|---|---|
| 3 | 1 | 2 | 3 |

After sufficient experience with the material the child knows that, after changing, there can never be more than one-less-than-the-base of any kind of block. (The material is confined to the four pieces described but it can be extended to *Long Blocks*, *Flat Blocks*, etc., as the need arises).

Once this is clear, a start can be made on the four processes. It is convenient to introduce addition with the aid of a number tray of five columns and five rows:

| LB | B | F | L | U |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |

Material is placed in the first two rows of the tray and the amounts recorded:

| B | F | L | U | (Box 4) |
|---|---|---|---|---|
| 2 | 1 | 3 | 2 | |
| 1 | 0 | 2 | 3 | |

The material is then collected into the fourth (answer) row when, in most cases, 'changing' becomes necessary. This being done, the answer is recorded, and the full sum reads:

| | B | F | L | U |
|---|---|---|---|---|
| | 2 | 1 | 3 | 2 |
| + | 1 | 0 | 2 | 3 |
| | 3 | 2 | 2 | 1 |

The actions involved show clearly that addition is a 'putting together', that 0 is a place-holder indicating 'none of' and that 'carrying' is a change of units and therefore of place.

The child makes up many similar examples from the same and from different boxes until the teacher feels that he is ready for the next step. This is to use coloured

counters or rings for the addition. The counters are set out on the tray in the same way as before and the identical actions performed. But here there is not the same visual check on place value. A prior decision has to be made about the value of the base and the order of the colours. A child may decide, for example, that 4 green counters are worth a yellow one, that 4 yellow counters are worth a red and that 4 reds are worth a blue. Ability to make this decision and act consistently with it shows that a certain degree of abstraction has been reached. (Appreciation of the completely arbitrary nature of the choice of base and order of colours must be achieved).

Subsequently the additions are performed without any concrete aid—but with a useful store of visual and muscular imagery.

Subtraction is much more difficult than addition to deal with—as all teachers know. To say that mathematically it is merely the operative inverse to addition is to obscure the fact that we use the same word to describe quite different mental operations. (At a later stage the use of one word and one symbol enables us to take for granted the mathematical equivalence of the different processes but if we insist on the equivalence before it has been perceived we will only lay up trouble for the future and make problems involving subtraction impossibly mysterious). It is not difficult to think of situations in which the operation is approached from three different points of view. They can be summarised as (1) taking away; finding a difference by (2) building up the smaller to equal the larger; and (3) taking away from the larger so that the smaller is left. We are all familiar with the 'counting on' response to a question which we have worded as if it were a 'take away' and this ought to be seen as the danger signal to us that we have used the wrong terms. We have used the terms appropriate to (1) to describe a situation which is intuitively seen to be of type (2).

As Dienes points out,[3] we are expecting a mature realisation of mathematical equivalence before the child has the necessary experience. With the material this experience can be given. The 'complementary addition' aspect is presented by asking the child to build two towers with the blocks and find the difference between the amounts of wood. In most cases the child will add wood to the smaller tower, but it can be suggested to him that he can also do it by taking wood away from the larger. The situation is made progressively more difficult: initially the bigger tower is made of *Blocks;* then of more pieces of each kind than in the smaller one; lastly with no restrictions.

'Taking away' examples are set out on the tray used for addition exercises and the child is asked to take away from the first row what he sees in the second. Again these are graded in difficulty, but the solution of a typical general example goes something like this: the material is set out and recorded:

| B | F | L | U | (Box 6) |
|---|---|---|---|---|
| 3 | 2 | 4 | 1 | |
| 1 | 3 | 3 | 5 | |

5 *Units* cannot be taken away from the top row unless more *Units* are put there. These can be obtained by changing one of the top row *Longs* into 6 *Units*. Now there are 7 *Units* in the top row. Taking away 5 *Units* leaves 2 *Units*. There are now only 3 *Longs* in the top row. Taking the number of second row *Longs* away leaves 0 *Longs*. Now there are not enough *Flats*, so one of the top row *Blocks* is changed into

6 *Flats*. Lastly 1 *Block* is taken from the 2 remaining *Blocks* in the top row. The completed example is:

| B | F | L | U | (Box 6) |
|---|---|---|---|---|
| 3 | 2 | 4 | 1 | |
| 1 | 3 | 3 | 5 | |
| 1 | 5 | 0 | 2 | |

This is seen to be a clear demonstration of the 'decomposition' method.

It is surprising that Dienes deals so inadequately with division after he has made this careful analysis of subtraction.[4] Division, in fact, is the next process to be taught in this scheme preceding multiplication. "It is easier to introduce division before multiplication, as the material is all on the table. All the child has to do is to *make* a certain number of equal piles."[5]

It is clear from these words that the word 'division' is taken as being synonymous with only one of its aspects. Since the trouble has been taken to devise questions involving 'taking away' and complementary addition, it would seem equally necessary to devise situations in which both 'sharing' and repeated subtraction are used. That these two aspects are completely different at a pre-mechanical level can be seen by considering the instinctive methods of solving, say:

(1) If I have 27 pennies, how many can I give to my three friends so that they each get the same number of pennies?

(2) If I have 27 pennies, how many of my friends can I give exactly three pennies to?

To a child who tries to solve these questions practically they will seem very different and it may be some time before he grasps their mathematical equivalence.

We also know that before a child can perform division sums economically he needs to have a good many multiplication facts at his fingertips. This would seem to support the necessity for handling multiplication first even if it is not such an easy concept.

Finally, the essence of the conventional method of long division is that it is a condensed form of repeated subtraction, the aspect of division which the cards do not deal with.

It appears to me that the Dienes scheme only half illuminates the *pure* mathematical concept and leaves the *techniques* of calculation used in the conventional notational context in almost complete obscurity.

But when we come to the notion of multiplication, Dienes has some extremely valuable and helpful things to say. The M.A.B. material only permits the putting of 'short' multiplication as a task to be performed, but within that limitation it shows (as with addition) the significance of place-value and carrying figures. When one moves on to compound multiplication, full understanding needs an appreciation of the associative and distributive laws of arithmetic. When a child considers an example such as:

$$\begin{array}{r} 97 \\ \times 35 \end{array}$$

he needs to know that $97 \times 35 = 97 \times 30 + 97 \times 5$

(i.e. $a(b + c) = ab + ac$: Distributive Law A)

and that $97 \times 30 = 90 \times 30 + 7 \times 30$

(i.e. $(x + y)z = xz + yz$: Distributive Law B)

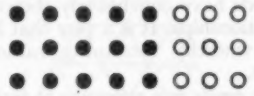
and that $7 \times 30 = 7 \times (3 \times 10) = (7 \times 3) \times 10 = 21 \times 10 = 210$
(i.e. $p(qr) = (pq)r$: Associative Law)

We may well have forgotten that all this is involved in calculations which have become for us completely mechanical. But Dienes is right to draw attention to the character of the underlying assumptions that enable us to perform calculations of this kind.

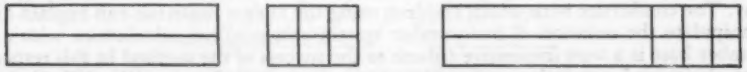
The associative law is approached by setting the task of building rectangular blocks with, say, 30 or 42 *Units* (the numbers being chosen to have three prime factors) and analysing the number of 'walls' or 'layers' and 'rows' or 'towers'. In practice, this exercise seems to confuse many children although it is not easy to conceive another task which would exemplify the law so well. Almost any direct concrete representation depends for its success on the ability of the child to understand volumes as measured by a triple product and this, as we know from other experiments, is not an ability which develops until 11 or 12, on the average. This seems to be a case where concrete realisation does not help and the law is best approached as a generalisation of experience with calculated triple products. (The valuable point, though, is that an attempt is made to put the associative law, as a tool for future use, in the hands of the child. Too often this is left to chance).

The M.A.B. material is not suitable for presenting situations from which the distributive laws can be abstracted. Dienes suggests that a simple balance graduated both sides from 1 to 9 and with hooks at the graduation points can be used effectively here. Exercises of the type: 'Put a ring on hook 5. Now put two rings on different hooks on the other side to balance this one,' are set. Trial and error or insight will lead to putting a ring on each of hooks 1 and 4 or hooks 2 and 3. The exercise is repeated with 2 rings on hook 5 and a demand that it should be balanced with 2 rings on each of 2 hooks. Again hooks 1 and 4 or 2 and 3 will be used. Further extension is made to more rings on each hook until the structure of the situation is grasped.

Following the principle of perceptual variability, the structure of rows of pegs on a peg-board is analysed. Each row is made of a fixed number of pegs of each of two colours. The total can be found by considering only the rows as wholes or by considering the colours separately. The array:



can be analysed as 3 rows of 8 pegs or the sum of 3 rows of 5 black pegs and 3 rows of 3 white pegs. i.e. $8 \times 3 = 5 \times 3 + 3 \times 3$. Varying the number of rows and the pegs in each row will give a glimpse of the validity of the distributive law. Another situation uses rectangular strips (S) and squares (units) and considers configurations like:



(I should perhaps point out here, in case it is confusing to anyone familiar with Dienes' own account, that I interpret the second term of a product as the multiplier. Using the other convention does not affect the argument).

With this background of experience, compound multiplication becomes less mysterious and less a matter of tricks passed on by the teacher.

**The Arithmetic Scheme**

It would seem a good moment, before briefly considering the Algebraic Experience Material (A.E.M.), to make a few comments about this scheme for teaching arithmetic.

First, it should be noted that Dienes does not claim that the M.A.B. and existing work cards provide all that is necessary for Juniors learning the four rules. He says, "It is taken for granted adequate *technical* facility will be acquired by children through the usual teaching techniques".[6] But he does claim that his methods will promote understanding of the concepts involved.

The method is attempting to do two things: to teach the *pure* concepts of the usual arithmetical operations (the nature of addition, subtraction, etc.) and to teach understanding of the way that *calculations* are performed in our conventional notation. If the Dienes material is compared with other structural materials, for example Stern or Cuisenaire, it is seen to be weighted on the side of the latter aim. This is simultaneously its strength and its weakness. Because of its inherent stress on positional notation it is not well adapted to concepts which are independent of notation. The commutativity of addition is not made immediately apparent as it is with the other two materials. I have already given another example in discussing the treatment of division.

The material is not geared to the development of number properties (prime, factorable, square, odd, etc.) nor to providing experience which will lead to familiarity with number bonds. The first omission is made good in the Dienes scheme by exercises with pegboards, balances and geometrical shapes; but the second is not. Number bonds have to be acquired some other way—presumably by the usual kind of experience and drill. If this is not done the child is forced to add and subtract by 'counting on' and 'counting back' in every situation that he meets. This, as the Stern and Cuisenaire methods have conclusively shown, is inefficient and obscures the structure of number relationships. It is a pity that Dienes does not admit to his scheme some similar device whereby 'measuring' rather than 'counting' is used to analyse number relationships.

The scheme surprisingly omits any reference to fractions.

When all this has been said however, it must then be said how successful this material is in some ways—spectacularly so in showing the real significance of our system of positional notation. This, it is fair to say (and is easily verified by anyone taking the opportunity to ask children at the beginning of the secondary stage a few searching questions), is frequently not understood in any but the most superficial way. The confidence with which children using the Dienes materials can explain and manipulate the notation of our number system and perform calculations with any number base is a very impressive tribute to the success of the method in this respect. The clarity and conviction with which they can explain the mysteries of 'carrying' and 'borrowing' in addition and subtraction sums I have not found elsewhere. It is

impossible to observe these consequences without wanting all children to acquire the same insights with the same ease.

**The Algebra Material**

The A.E.M. consists of various flat pieces of wood in the shape of squares, rectangles and equilateral triangles, a graduated balance and rings as described earlier, pegs and pegboard. It is usually supplemented with a set of small boxes and any suitable small objects for them to contain. The work cards cover among other things the laws of arithmetic, perfect squares, use of brackets, linear equations, simultaneous equations, directed numbers, factorisation of trinomials and solution of quadratic equations. It is not practicable to discuss all of these things here and I shall limit myself to outlining part of the treatment of just one or two topics.

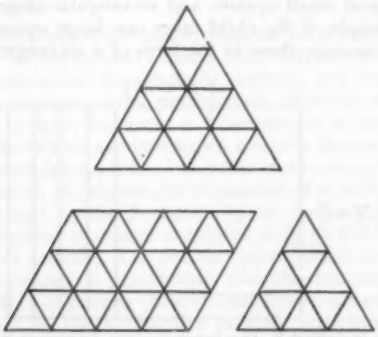
The concept of a square is approached by instructing the child to take a random number of small squares and see if he can make a larger square with them. When he has experimented sufficiently he records the numbers of small squares which he needs. This is then repeated by taking a random number of pegs and trying to form a square on the pegboard. In order that the concept shall now be divorced from the shape, he is set a similar task with small equilateral triangles and with small diamonds (formed from two of the triangles). Removing the geometrical aspect entirely, he now has to put objects into boxes so that each box contains as many objects as there are boxes. All these experiments produce the same set of results. (An unfortunate side-effect which I have occasionally noticed is that a child will calculate the number of small triangles in a larger one by multiplying two adjacent sides—which works in this context!).

The factorisation of a perfect square is introduced by splitting a square made of small squares into four parts and recording the result:

e.g. $8 \times 8 = (3 \times 3) + (3 \times 5) + (5 \times 3) + (5 \times 5)$

This is repeated with triangles. The following figure illustrates

$$7 \times 7 = (4 \times 4) + 2 \times (4 \times 3) + 3 \times 3$$



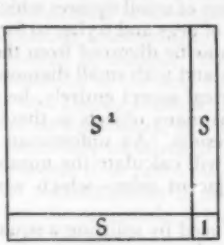
(The middle term is obtained from the parallelogram part by considering the triangles in two groups—vertex up or down). A similar experience is obtained by

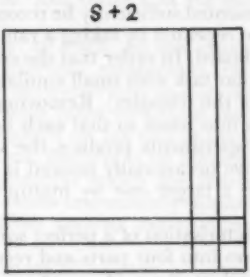
putting, say, six objects into each of six boxes, dividing the boxes into two groups of two and four boxes and removing from the boxes in each group enough objects to leave the same number inside as there are boxes in the group. In this case four objects are taken from each of the boxes in the first group and put by the side of the boxes. Similarly two objects are taken from each box of the second group. Analysing the number of objects initially and finally gives:

$$6 \times 6 = 2 \times 2 + 4 \times 2 + 2 \times 4 + 4 \times 4$$

| original arrangement | inside two-box group | outside two-box group | outside four box-group | inside four box-group |
|---|---|---|---|---|

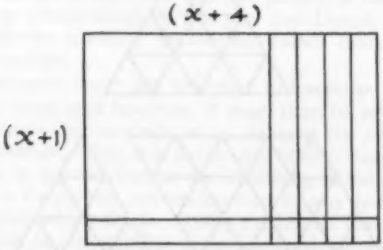
Following upon this, tasks are given which build up squares and triangles of increasing size. A square which has a side called $a$ (and is equivalent to $a \times a$ small squares) can be built into a larger one by the addition of two rectangular strips and a small square, or from four rectangular strips and four small squares, etc.




These illustrate the identities $(a + 1)^2 = a^2 + 2a + 1$ and $(a + 2)^2 = a^2 + 4a + 4$ and lead eventually to the basic identity $(a + b)^2 = a^2 + 2ab + b^2$.

Use of the large and small squares and rectangular strips leads to factorisation of trinomials. For example, if the child takes one large square, five strips and four small squares, he can arrange these in the form of a rectangle.




If the length of a strip is called S, the large square becomes $S^2$ and the total area of the rectangle is $S^2 + 5S + 4$. But this is also a rectangle with sides $(S + 1)$ and $(S + 4)$. This constructive approach to factorisation can be extended to cases with

a negative middle term by using the strips to cover (instead of add to) the area of the other pieces, and to cases with a negative final term by using the device of additional strips in pairs, one to add to the area and the other to cover up. Readers might like to try for themselves the factorisation (by these devices) of $S^2 - 5S + 6$ and $S^2 + S - 6$, solutions to which are given on Pp 96, 98 of "Building Up Mathematics".

The principle of perceptual variability demands that these exercises be repeated with, say, pegs on the pegboard.

Linear equations are developed from the balance, strips and small squares, triangles of two different sizes and pegs and pegboard. As a very elementary example, consider the solution of $2x + 1 = x + 5$. With the balance, a ring is put on hook 1 on the left and another on hook 5 on the right. Then the child has to find a hook number so that putting two rings on the left hand one and one ring on the right hand one will effect a balance. Or he puts two strips and one small square in one pile and one strip and five small squares in another and changes the strips into the appropriate number of squares so that both piles have the same number of squares. A similar method works with the triangles and with rows and single pegs on the pegboard.

An extension to simultaneous equations using strips of two different lengths can be easily seen.

**Conclusions**

This, then, is a taste of the Dienes scheme. The question which one must ask is whether it is more than just a collection of devices, some old, some new; some good, some bad. In spite of the fact that the scheme is not complete and is under constant revision it is possible to see that it has a unity of approach which derives from the fundamental principles summarised earlier. Whilst realising that we have not sufficient evidence as yet to confirm or condemn these principles with any confidence I would take issue with one of them.

The principle of perceptual variability rests on very shaky ground. Dienes puts forward two claims for its importance: that a child may more easily see the structure of a situation in one perceptual form than in another; and that abstraction involves isolating the common elements from perceptually different situations. The first of these two claims may be true, but since it is impossible in practice to devise a very wide range of situations which are structured towards the same concept and are of approximately equal complexity, it has led to the inclusion in the scheme of exercises too obscure to be effective. (Compare the expansion of a perfect square by means of splitting up a square and by the box-and-objects method. The latter is too 'unnatural'. The mathematical structure is related only to the final recording and in no way to the necessary actions). The second claim (which seems to be regarded by the author as more important) is superficially plausible but is not as true as it sounds. It assumes that the effect of putting different successive situations in front of a child is to make him attend to the similarities in their structure. Because his natural interest is in differences rather than likenesses this, in general, he does not do. Structural similarities have to be sought and the ability to look purposefully for them comes with maturity, through experience and, possibly, training.[7] In fact the presenta-

tion of perceptually different forms quite frequently leads to confusion. Where it does not do so one can be reasonably sure that a successful abstraction (or discovery of structure) has taken place in one of the situations and that this insight is then being applied to the others. In short, the abstraction must have taken place *before* the perceptually different forms become meaningful.

I believe that the hypothesis of the desirability of perceptual variability runs contrary to the available evidence and that the Dienes scheme would be considerably improved by dropping it. This need not result in a diminished richness of experience as it could be replaced by a new hypothesis that required each new insight to be applied in a variety of situations. The new hypothesis would remove the need for the presence of some exercises whose only justification is that one must find a different perceptual form (or bust) and would put a further emphasis on the need for *using* the mathematics that is being discovered.

The principle of mathematical variability is another matter. The stress put on this is one of the more valuable aspects of the scheme and will have the effect of making many teachers inspect their work at any level to see if they are successfully creating appreciation of the full generality of mathematical concepts. One need not though, fall into thinking that a mathematical concept necessarily emerges Minerva-like and ready-for-anything on first acquaintance. Here and there the scheme appears too exhaustive too soon.

The great strength of the experiment is in the kind of tasks which, with skill and ingenuity, have been devised. Learning springs from doing. Concepts are developed from experience. This approach to mathematics teaching, as unlike 'chalk and talk' as possible, is based on secure learning principles. Those who react by saying that *they* learnt without this concrete experience—and this is the regrettable reaction of many teachers—overlook the large numbers of their contemporaries who were not so lucky. The most convincing testimony to the method comes from the teachers using it who admit frankly that their eyes have been opened and their understanding of mathematics enlarged. If it achieved nothing else it would be worth doing for this alone.

It is impossible to cover an experiment such as this in enough detail for those who are not familiar with it to be in a position to judge it. On the other hand, our professional interest demands that we know that such experiments are taking place and something of what they are trying to do. I hope that what I have said sufficiently indicates that this is an important experiment which is going to have an influence on methods of teaching mathematics. What form the influence will take is not yet clear. It may be very specific and involve very many more schools in this scheme. It may be more diffuse and general, gradually effecting changes in approach in schools which do not take over the method in its present form. At the very least it is one more piece of evidence of a steadily increasing tendency for the teaching of mathematics to break away from the dead hand of formalist treatment. This is a trend which we dare not (cannot) resist.

*REFERENCES:*

1 See the figures quoted in the article by W. A. Brownell in "The Arithmetic Teacher," April, 1960. Also by J. B. Biggs in "Educational Research," February, 1961.

2 No results of the Action Research in Structural Methods undertaken by the National Foundation of Educational Research have been made public yet. The report, when it comes, should add to our knowledge of the effects of structural materials.

3 See pp. 62-69 of "Building Up Mathematics" for a fuller analysis of this matter of subtraction.

4 Some cards (Set 2G) in the A.E.M. scheme deal with the two meanings of division but do not meet the above criticism. The concept of division as repeated subtraction is not developed.

5 P. 69, "Building Up Mathematics."

6 P. 66, "Building Up Mathematics.".

7 Some support for this view comes from "Thinking" by F. C. Bartlett. See Pp. 93-96.

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O.E.E.C. have recently published a report on a seminar they held at Ashridge (England) in July, 1960 under the title *Television for School Science*. There is little direct reference to the teaching of mathematics by television except for an account of an experiment which was to be held (it has now taken place) in the Lille area of France. The experiment was intended to cater for three situations (*a*) where a teacher was available who could view with the class and then give further work on the subject matter, (*b*) where the class viewed with a supervisor or non-specialist teacher, practical work being done later by the pupils, and (*c*) where isolated pupils would be able to supplement work sent by organisers of correspondence courses. The work was mainly for 12 year-olds and covered *surface* and *volume*. The report as a whole may be of interest to anyone particularly concerned with the application of television to teaching.


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# SQUARES OR SECONDS?

C. J. GOOLD

Sometimes difficulties are created for children, particularly the weaker ones, if when new ideas are being discussed, the teacher cannot resist the temptation to display, during the discussion, the classical method of meeting the situation. However hallowed the tradition, it may in fact contain unnecessary sources of confusion and if so, it might well be left until later.

There is also the disadvantage that every time the children 'discover' some mathematical relationship, they will expect the inevitable anticlimax to follow immediately,— a reminder that it has been done before by what they will come to regard as an 'authority'. What mental picture do they have of these 'authorities'? Do they see them as we want them to see them, people like themselves experimenting and making mistakes, or do they form a conception of 'authorities' with a complexity and a sense of completion like they may have about Monarchy or Divinity? Teachers are not immune from the authoritarian view, often because of the way in which they themselves were taught. In geometry, for example, a situation was provided, followed by the command:- *Prove* $AB = CD$. This, of course, implied that the 'master' had already found it to be so, and the task of the disciples was to try their skill at following his footsteps. Would it not be better for us now if we had spent our years of geometry answering the question:- *Is* $AB$ *equal to* $CD$?, a process which might have encouraged exploring, instead of bringing it to an end?

If this is accepted, one can doubt many things that are still done. For example, one can doubt the wisdom of associating $3 \times 3$ with a square, and of actually calling it 3 squared. Similarly $3 \times 3 \times 3$ is associated with a cube and always called 3 cubed. Thus the powers of numbers are firmly tied at the outset to these mental models. What happens to the whole idea when we come to $3 \times 3 \times 3 \times 3$? Undoubtedly a difficulty has now been created for the children, because they can form no mental model. What do we do about this difficulty, or do we think it does not exist? Might it not be better to forget the terms 'squared' and 'cubed' so as to break the inevitable and misleading association with models?

We can say, not later, but from the beginning,

$3^1 = 3$ to the first power
$3^2 = 3$ to the second
$3^3 = 3$ to the third
$3^4 = 3$ to the fourth
$3^5 = 3$ to the fifth

and there is then no difficulty in passing from 3rd to 4th.

Instead of these usual models, which introduce areas, volumes, and the whole idea of dimensions, it would seem more appropriate at this stage to illustrate powers by patterns.

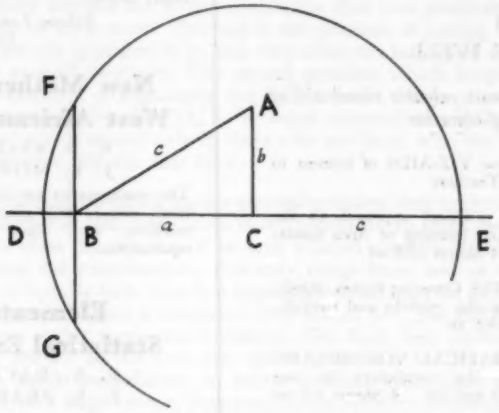
e.g. $2^1$ * *

$2^2$ * *
* *

$2^3$ * * * *
* * * *

$2^4$ * * * *
* * * *
* * * *
* * * *

Again, having firmly fixed the second power of a number to a square, the temptation is to follow as quickly as possible with Pythagoras' Theorem, a temptation seldom resisted. Now if squares and cubes are to be avoided elsewhere, they should be avoided here also, which means leaving the traditional exposition until later, and finding a proof of the theorem which does not involve areas. How often is this done? I would suggest the following:-




ABC is the right-angled triangle.

With centre C, radius $c$, a part-circle is drawn.

BC produced cuts the circle at D and E.

A perpendicular is drawn through B, cutting the circle at F and G.

i.e. $DB = c - a$
$BE = c + a$
$FB = BG = b$

Then, from the intersecting chords DE and FG,

$$DB.BE = FB.BG$$
$$(c - a)(c + a) = b \times b$$
$$c^2 - a^2 = b^2$$
$$c^2 = a^2 + b^2$$

I wonder how many teachers would be offended if this result were read, not in the usual way, but as:-

$c$ to the 2nd $= a$ to the 2nd $+ b$ to the 2nd?

However, this is how it should be read, for it has nothing to do with squares.

ULP

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# SECONDARY MODERN SCHOOL MATHEMATICS — VII

## G.C.E. AND EXTERNAL EXAMINATIONS

CLAUDE BIRTWISTLE

Meeting Secondary Modern School mathematics teachers at courses in various parts of the country has led me to the conclusion that two problems predominate in the organisation of their work. The first is the problem of having teachers taking the subject who are not qualified in it, and very often not interested either, but have to take it owing to staff shortages. The second problem which frequently arises in discussion is the question of organising the mathematics course when a number of pupils are being entered for the G.C.E. or other external examinations. This is a 'bread and butter' article aimed to help those who are faced with the latter problem, some of them, perhaps, for the first time.

The most recurrent questions on this general problem deal with the organisation of an examination course within the limits of the timetable. In small, or even medium-sized, schools it is clear that there will be only a small number of pupils each year who will be sitting the examination; this may range from two or three pupils to a dozen or so. It is equally clear that it is impossible to arrange for so small a number to be taken in separate classes throughout their school life, and the problem is how to cope with them under these circumstances. The fault here probably lies in the approach of the teacher to the fact that there is to be a G.C.E. course, and the teacher tends to think immediately in terms of what takes place in a grammar school. Here, to a very large extent, the examination syllabus determines the syllabus of the school. However, there is no reason why this should be so, as is being shown increasingly in many grammar schools where a more progressive outlook prevails. It is for the teacher to determine his own syllabus according to his own ideas, and once he has the courage to do this a solution to the problem of the small group presents itself.

The teaching of mathematics at the secondary level should aim at the understanding of basic mathematical principles and their application, and this is the same for all pupils, whether taking an external examination or not. It should be possible, therefore, to draw up a basic curriculum for the first three years of the secondary school course which aims at doing just this without any thought of what is to be found on an examination syllabus. If this is designed for the 'A' stream of the school, it is possible to select the better pupils at the end of the third year for more individual tuition during their last two years at school. It may not be possible even for the school to afford the luxury of having a teacher take a class of half a dozen or so to G.C.E. standard for two years, but a compromise may be affected here also by taking the fourth and fifth year G.C.E. candidates together as separate sets. This, of course, calls for careful timetabling.

Reverting to the first three years, it is quite likely that there will be a wide range of ability within the 'A' stream class and an attempt to teach the class as one unit may hold back the brighter pupil who is destined for G.C.E. work. One

solution is to arrange the class in sets taking the same topics to different levels, while another solution is to arrange for more individual work by introducing a system of work cards. Whichever method is chosen, it is possible to have the whole class dealing with one topic at the same time, but with a wide variety of treatments suited to the ability of the pupils. No doubt in all cases there would be a practical approach to the subject by way of experimental or discovery methods; this could be followed by class discussion and further practical work. It is at this stage that one would look for a deeper study by the more able candidates. Thus, whereas a demonstration of a geometrical theorem might suffice for the larger part of the class, the more able ones would now look to something in the way of theoretical proof. Similarly, following on this one might expect more difficult and academic types of example for the more able ones, while the rest are still working examples of a practical nature. The introduction of homework for the more able pupil also helps in this type of organisation. No attempt should be made, however, to discard well tried methods in the secondary modern school in the belief that these are inappropriate for the more gifted pupil. It should be remembered that what are good methods for the less able also help the brighter pupil to an initial understanding; in fact, many grammar school teachers could learn quite a lot from studying teaching methods employed in some secondary modern schools. What is new is that in the case of the brighter pupil the teacher has to develop these methods so as to give a fuller treatment of the subject matter.

The main point, however, is for the teacher to think clearly about what mathematical *principles* he wishes to include in his syllabus rather than what type of *examples*. There are, in fact, surprisingly few mathematical principles taught in the normal secondary school course. As an example one might take proportion. Proportion underlies the greater part of the arithmetic and algebra usually taught in the secondary school; many types of problem frequently taught as separate topics are just particular examples of this general principle. The reader is left to consider this further for himself; he may be surprised at the result!

In the last two years of the course, and especially during the last eighteen months, it is very desirable that the G.C.E. group is taken separately and not mixed in with a larger class. This latter could not occur in the fifth year, of course, since only those staying on to take the G.C.E. will still be at school. This should enable small groups from the fourth year to be taken at the same time as those from the fifth, as has been pointed out above. With groups of up to ten or twelve it is possible to have two independent groups working in the same classroom, and provided the work is planned with a little thought there will be no wastage of time by one group while another group is being taught. It is during these last two years that more attention can be given to the detailed syllabus of the examining board. But the aim should not be to teach to the syllabus; rather it should be to fill the gaps in the earlier part of the course, to give a deeper understanding of the subject, and to make certain that the pupil is able to apply the principles he has learned to the various topics of the syllabus. Perhaps a word of reassurance should be given about taking groups within a class or individual work. Many who have not done this sort of thing are put off by the thought of it. In fact, the main essential is to think the matter out clearly and plan it. The initial planning and organisation takes most work, but if this is done effectively there can be little trouble later.

It goes without saying, of course, that throughout the five years there must be continuous and progressive revision and a thorough revision is looked for in the last twelve months of the course. A good revision book can be invaluable here, but revision should be interspersed with new work; nothing can be more uninteresting and soul-destroying than a long period of continuous revision.

While reference has been made specifically to G.C.E. work in what has been said, the same applies to the same or a lesser degree, to other types of examinations, and as the general move seems to be for an increased number of examinations to be taken in Secondary Modern Schools, many teachers in these schools should think very carefully of how they are to plan their work to meet the situation. Many pupils already in lower forms of schools which do not at present take examinations are going to be entered for external examinations before they leave.

To return to the G.C.E. it may be worth mentioning for the benefit of those unfamiliar with the examinations that the different Examining Boards display a variation in the type of papers they set as well as in the syllabuses, although the standard is roughly the same in all cases. Thus some Boards favour a paper with a more practical flavour, some place more emphasis on trigonometry at the expense of geometry, some introduce questions on geometrical drawing and so on. This, of course, is in addition to the two syllabuses, A and B, which each Board offers. Before deciding on any particular Board, therefore, it is worthwhile to study the syllabuses and past question papers of various Boards to see which best suits the particular requirements of the school.

In the case of all the Examining Boards certain points are looked for by examiners and teachers would be well advised to bear these in mind throughout all their work. The first essential is the ability to think out a problem logically and apply the appropriate mathematical principles to the situation. Clarity of expression and careful setting down are necessary, while computational accuracy must be a prime feature, of course. With regard to the latter point, the need to estimate and check answers should always be insisted upon; one may forgive mistakes, but unreasonable answers are inexcusable.

Finally, let it be emphasised that external examinations need not have a restricting influence on the work of a school. The primary aim of the mathematics course should be the understanding of principles. Provided these are understood in the earlier part of the course, the latter part can be devoted to more detailed coverage of the topics on the examination syllabus. To attempt to devote five years to training in the techniques of examination questions is to court disaster both in the mathematical development of the pupils and in the examination results themselves.

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Would readers make a point of mentioning *Mathematics Teaching* when replying to advertisements in the magazine? This is both helpful to the publishers and of service to the magazine.

## NOTES AND NOTIONS

One of our readers has written to us giving details of a novel idea for drawing graphs on an ordinary blackboard. It consists of a roll of plastic material on a cylindrical wooden rod mounted above the blackboard so that the material can be pulled down in front of the board when required and automatically rolled up again, like a sunblind, when not. In the material are a series of small holes spaced at equal distances in a square pattern, corresponding to the squares of the graph paper. To use the apparatus one merely rolls it down and rubs over it with the blackboard duster, after which it is rolled up. On the board are a series of dots which serve as a graph network. Our reader does not say where the material has been obtained, but the material used for large cinema screens has a pattern of holes of this nature.

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The Manchester University School of Education are holding a one-day conference on Saturday, 2nd December, 1961 on *Modern Mathematics*. Among the speakers will be Prof. Papy and Dr. E. A. Maxwell. Full details may be obtained by writing to Mr. W. G. A. Rudd, Organising Tutor, School of Education, The University, Manchester, 13.

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Pictorial Charts (153 Uxbridge Road, Hanwell, London, W.7.) have just published a chart in their series, which forms a useful display for a mathematics room. It is called "Mathematics and Progress." The chart reference number is W 101.

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"Was there more than one Homer?" This seems an odd question to appear in a mathematical magazine, but Mr. McDonough, a classics instructor at St. Joseph's College, Philadelphia, U.S.A., has attempted to solve the problem by mathematical means — he has used a computer. Mr. McDonough has scanned "The Iliad's" 15,693 lines and has used a metrical code he has devised to translate his results into numbers. Number one, for example, would be for a long syllable at the beginning of a word of at least two syllables; number two would be for a long syllable within a word of three syllables, and so on. In the end, each line is represented by a seventeen digit number. "The Iliad" had some 112,000 words and the machine's job was to sort out the words into 157 metrical types so that Mr. McDonough could determine if the style was consistent throughout. He used an International Business Machine Corporation's "650" Computer and it took five hours to analyse Mr. McDonough's data. Without its aid he would have had to write out 112,000 index cards, sort them and file them into 157 boxes, then count and copy the contents of each box. We understand that Mr. McDonough plans to start work on a similar project soon using "The Odyssey"; once again the machine will help to solve the problem of whether it was written by one man or more. Somehow there seems to be endless possibilities here: what about those thirty-two exercise books which were

handed in bearing thirty-two indentically worded geometrical proofs? And one wonders if Durell really did write *all* those textbooks (offers to scan Mr. Durell's works should be sent to the Secretary of the Association and *not* to the Editor!).

Oh! and by the way, it took Mr. McDonough four years to prepare his material for the computer. And what did the computer say? It said it rather appeared as though Homer had written the lot. Pity Mr. McDonough didn't believe what the publishers printed on the cover!

---

*Cue up!* This is the title we are giving to a problem sent to us recently by Mr. D. E. Winks of Camberley. This is not suitable for our usual puzzle feature since it is of a difficult nature and there may be more than one way of approaching a solution. Those of our readers who do try the problem and obtain a solution are invited to send it to the Editor for possible publication in our next issue. Given a circular billiard table with perfectly elastic cushions and two balls A and B anywhere on the table, construct the path A must take to hit B after one reflection (i.e. the problem is to find the point, P say, on the cushion, at which A must be aimed to rebound to hit B).

---

## APPARATUS REVIEW

### Viz-Aid.

| | |
|---|---|
| Set No. 9 | Area 12/6d. |
| Set No. 10. | Fractions. 12/6d. |
| Set No. 11. | Mathematical Vocabulary. 7/6d. |
| Backcloth. | 25/–. |

Designed by Arnold and Jacqueline Ivell. Published by University of London Press.

This is a development of the principle behind methods known to teachers for some time; it uses the self-adhering property of certain materials as in the Flannelgraph method. The backcloth is a black PVC plastic 30″ x 40″ mounted on rollers and the models are card backed with plastic which can be placed in position where they remain until no longer required.

Set No. 9 *Area*, contains firstly a series of strips, rectangles and squares, in particular 37 squares 1″ x 1″, enabling the concept of area as "space coverage" to be demonstrated. It is easy to show that area is independent of shape by moving pieces around ad lib. Six dissected squares of one inch side are provided. Four divided along the diagonal and two into quadrants of a circle. (Why not four?) With these pieces a wide variety of illustration is possible. The set also includes some irregular shapes and a triangle, parallelogram and trapezium with dissected copies to show the equivalent rectangles and these are likely to be as useful as models of this kind can be. There are also a set of coloured squares, complete and dissected, to illustrate Pythagoras (two methods).

Set No. 10 *Fractions*, provides several groups of models each consisting of a "unit" and a number of fractional parts. The "unit" shapes are: strip, circle, rectangle, equilateral triangle, hexagon, and the fractions used are those which give a reasonably easily recognised relationship, e.g. strip has halves, thirds, quarters, sixths and twelfths, while the triangle has only thirds. The "units" are all white and the "parts" are both white and coloured so that pieces of the same colour represent the same fraction, although the converse does not hold. This set provides for the conventional demonstration of fractions as the relationship of whole and part, and by offering a variety of unit models ensures that the children do not always associate a particular shape with a particular fraction. Equivalence of fractions, and addition and subtraction can be demonstrated with this set.

Set No. 11 *Vocabulary* is a set of words (nouns and adjectives) and numerals, in white on black. Lower case letters are $\frac{3}{4}$" high and numerals $1\frac{1}{2}$" high. It attempts to provide everything which a teacher is likely to need in whatever way he uses sets 9 and 10. The selection of 49 words (with 's' 10 times for plurals) is certainly comprehensive and although any teacher can probably think of something he would like to see included there is nothing entirely redundant. Numerals from 1 to 12 are provided with a bar beneath for numerators, (e.g. $\frac{8}{}$) and 2, 3, 4, 6, 8, and 12 for denominators. More of these, particularly plain, would be an advantage.

With this material children can take part in a lesson more conveniently than would be possible if the same work were done on a blackboard, but it is mainly for the teachers' demonstrations. The pieces are convenient to handle and likely to stand up to reasonable use in the classroom. Adhesion is good, pieces remaining in position as long as required excepting in a strong cross breeze! A teacher using the material regularly will probably find it desirable to mount the backcloth on a board and with such a rigid base even these adverse conditions might be withstood. A drawback is that pieces cannot be superimposed as the cards do not adhere to one another, and this was found to be a real disability in practice, particularly when using the fraction set.

The publishers should perhaps look again at the bags containing the sets. The legend on No. 9 includes the following:-
*Triangles.*
*1 right angle (for Pythagoras' Theorem)*
*1 large scaling triangle.*
*1 dissected congruent triangle.*

D.T.M.

## LETTERS TO THE EDITOR

Sir,

Is $\pi$ three and a bit or four times something? In a sense the answer depends on what we mean by $\pi$ and the question is trivial but posing it may contribute to discussion of the important issues raised by Mr. C. T. Stroud in a recent letter (Mathematics Teaching 16 p. 57). His distinction between modern and conventional mathematics hinges on the common error of supposing that mathematics is a cumulative discipline. The student of mathematics is envisaged as climbing a ladder

made of indispensable rungs. (A set of consecutive rungs is called *a syllabus* or *the Emperor's clothes*. School and University syllabuses are disjoint sets.) Somehow this metaphor carries the idea that the objects of mathematical study at the top of the ladder are different from those at the bottom. (cf. Prof. Thwaites and the 'Himalayas' recently publicised for the benefit of top people and lively minds.) But when the 18th century juggled the odd numbers into a form that represented one quarter of $\pi$ it was the same $\pi$ that Archimedes trapped between 3 1/7 and 3 10/71.

The lamentable misunderstanding of the purpose of Euclid's Elements makes us forget that Euclid was attacking the same problems as Hilbert. Similarly the distinction between the Sophists, who were the topologists of the 5th century B.C., and their successors today is not one of raw material but one of tools and techniques. It may well be that we must practice on flint heads before we can use a chisel but the distinction to be drawn is not between modernity and convention but between the quick and the dead. As for a technology that uses dead methods . . . But Mr. Stroud gives the game away when he refers to conventional work as being necessary for technology *as well as examination purposes*. (Both needs would be served by an O-level question that asked the student to programme some imaginary computor to solve the quadratic equation $3x^2+5x-1 = 0$. Little is served by asking for a formula solution alone, cf. Norman Douglas: *South Wind* p. 37).

Why does the fifth power of a digit end in the same digit? One generalisation of this phenomenon leads us to Fermat's theorem which is in itself a special case of Lagrange's theorem in the theory of groups. Too difficult for schoolboys? Who knows? Galois' schoolteachers thought they did. In any case the touchstone of 'modern' mathematics is its simplification of structure and its unifying of isolated phenomena and the ability to see the wood for the trees is most certainly 'needed at present for science and technology as well as examination purposes.'

Yours, etc.

D. G. TAHTA,
Dept. of Education, University of Exeter.

Sir,

I appreciate the difficulty of reporting concisely the proceedings of a Conference as lively and fruitful as the 1961 A.T.A.M. Conference seems to have been, and congratulate Mr. Fielker on communicating (*M.T.*, July 1961, pp. 34–52) so much in the way of information and atmosphere; but there are a few points on which I should like clarification.

1. "We could do no worse than we are already doing" (D. H. Wheeler, p. 52). Mr. Wheeler had earlier—p. 39—emphasised the importance of humility; but need we grovel?

2. "Algebra, as generalised Arithmetic, was rigid and useless." (G. Beaumont, p. 38).

3. "The algebra syllabus must include theory of sets, . . . groups, rings and fields; vector algebra; transformations; matrices; homomorphisms and isomorphisms." (C. Hope, p. 44). I take it this is with an eye to the subsequent University study of Mathematics. Is there evidence that British University and Technical College

Departments want this done, and to what extent? It isn't done in the U.S.S.R., if that matters.

4. "The use of numbers and their group properties plays a big part in selecting a number system and its algebra to serve as a model for a particular physical system" (C. Hope, p. 44). The group with respect to addition did not enter the number system until negative numbers were invented, some time after man began to weigh and measure. Obviously I have missed the point.

5. "It was impossible to develop advanced mathematics without the concept of function . . . Hence the idea should be introduced at the first moment of appeal" (G. Papy, p. 50). Does this really follow? (cf. Mr. Hope's "example of an early use of homomorphic theory" on p. 45. Perhaps the point is simply that the teacher should know the "important theorem" which subsumes the "many applications." Compare also the remarks of Prof. M. H. A. Newman on p. 170 of the *Mathematical Gazette*, Oct. 1959: "He [Max Beberman, of the Illinois project] may, as he says, make school-children understand that a relation is just a class of ordered pairs, but if so he is teaching them mathematical logic without first teaching them the mathematics that is its *raison d'etre*. This seems an unnatural way to go to work".)

And two points of detail:

6. p. 50, Do we really *prove* concurrence theorems by the application of set theory? Or is it simply that we express the familiar argument in terms of set notation (which may of course be worth doing for more reasons than one)?

7. p. 51. The symbol $P(48 \cap 36)$ is meaningless; and we cannot have "an example of distributivity" involving only the two elements P(48) and P(36). There is, of course, the usual double distributivity of $\cap$ and $\cup$, which leads to results like "The H.C.F. of (48 and the L.C.M. of 36 and 64) = the L.C.M. of (the H.C.F. of 48 and 36 and the H.C.F. of 48 and 64) if we feel it is worth the trouble.

We have much to ponder.

Yours, etc.

Alan G. Sillitto.

Renton, Dumbarton.

Sir,

It is not for me to reply to most of the points made in Mr. Sillitto's letter, but may I thank him for his kind opening remarks and answer two of his later ones. Mr. Wheeler, I believe, had his tongue in his cheek when he said, "We could do no worse than we were already doing"; an exclamation mark was the most I could allow myself in order to convey this impression. As for Prof. Papy, I said that "I was not altogether happy about some of his examples" (p. 32), and Mr. Sillitto's last two points were indeed those about which I was least content, but thought it just to include them in a report.

Yours, etc.

David Fielker.

Abbey Wood School,
S.E.2.

## BOOK REVIEWS

### On Teaching Mathematics

Report of the Southampton Mathematical Conference, 1961. pp. 116 + xxiv. Pergamon Press 7s. 6d.

The Conference is over, having achieved its aim of having its members write a book in nine days—the "Nine Days' Wonder." It hit the headlines, its Chairman appeared on television (and failed to answer "the 64,000 dollar question" from Mr. Michelmore!), the report was rushed through the press, and is now available for all to study. What has been achieved?

The book falls into two parts: the first, a report of the deliberations of the Conference; the second a report of the main lectures. It is the former which is of immediate concern to us. The *Introduction* is concerned with the shortage of mathematics teachers, and this theme is repeated in the second chapter, *Mathematics in Use*, where the so-called Industrial Committee reviewed the demand for mathematicians in industry and basic qualities looked for in industrial mathematicians. The third chapter concerns itself with the transition *From School to University*. There seems little here which has not been said elsewhere before, whether the drafting committee are considering the student's attitude to mathematics, methods of University entrance, facilities for teachers to keep up to date, or the syllabus in the Sixth Form. The latter, in fact, is discussed again in the following chapter, but when one studies the detailed syllabus there is little apart from the "Theory of Sets and/or Matrix Algebra" which does not appear in the Advanced level syllabuses of one or other of the Examining Boards. One very sound point which is made is that by avoiding overlap of work in Physics and Applied Mathematics it should be possible to find time to introduce more Statistics into Sixth Form studies. There is a useful, if brief, review of basic concepts that should form an essential part of early mathematical education. The chapter on *University Mathematics* looks to a wider variety of courses being made available for mathematics students and more attention being paid to students who are not taking a special honours course.

No doubt the conference was stimulating to those who took part, but the report conveys little of this and it is difficult to imagine it having any significant effect on the future teaching of mathematics in this country. Any move to reform must come from further investigation and discussion, and there seems no provision for this. Nor does it put forward or discuss any significant new ideas; many of its statements are, in fact, unsubstantiated.

Successful teaching of any subject depends on the methods of teaching employed, the background of the teacher and the content of the course. This report deals particularly with the last, introduces the second insofar as it touches upon this last, and makes little mention of the first. Nevertheless, the book should be studied by all teaching the subject in selective secondary schools and Universities in the hope that it might stimulate them further to think about the present crisis in the teaching of mathematics.

This volume is the first in a series under the general title *The Commonwealth Library of Science, Technology and Engineering* in which the publishers hope to produce

a thousand books in the next five years, covering levels from Secondary to University Honours. The book is well printed and for economy of production is issued as a 'paperback'; however, we regret to say that before we had finished reading it the back came off.

C.B.

## Mathematics in Your World

Karl Menninger | Bell, 1961 | 244pp. | 21s.

This book has been translated from the German and, while the actual translation is good, it is perhaps unfortunate that the translators have not chosen to Anglicize many of the examples. For instance, one finds examples in tonnes per metre (p. 40) where one might be more used to foot-pounds or foot-tons, and the well-known ready-reckoner dial of a shopkeeper's scales would have been better shown in lbs. and ozs. and shillings and pence (p. 127). Worse still, at times they do try to do this half-heartedly, as for example when discussing the lateral pull on a car turning into a curve. After long discussion in metric units, we read that it is rather as though the car were suddenly pulled by a force "of about 8 cwt."

However, the book is well thought out and original in its approach. The author is concerned with every-day phenomena which have a mathematical background, but they are introduced in such a way that one becomes interested in the problem and its solution before one realises what one is doing mathematically. The mathematical principles, in fact, are introduced almost apologetically; one has grasped the principles long before one is told that this is a G.P. or whatever it happens to be. This indeed is a book for the pupil who has no great liking for the subject, as well as for the enthusiast. At all times it is interesting, stimulating and intriguing. Everyone can find something of interest, whether it is about streets and highways, a flutter on the 'pools', life assurance, or many other topics which are used to show the importance of mathematics "in your world".

This book should be in every school library, and on every mathematics teacher's bookshelf.

C.B.

## A Course in Pure Mathematics

Margaret M. Gow

The English Universities Press Ltd. | pp. 619 + xi. | 40s.

The Purpose of this book is primarily to cover the new syllabus in Mathematics for Part I of the London B.Sc. General degree, assuring knowledge of A level mathematics. However, sufficient lead in to the various topics is given in most cases to allow the reader to revise his previous knowledge.

The contents comprise 25 chapters: Polynomials and Partial fractions; theory of equations and inequalities; determinants; limits and infinite series; binomial,

exponential, and logarithmic series; complex numbers; summation of series; differentiation and integration; coordinate geometry of straight line and conics, plane, straight line, sphere and quadrics; partial differentiation; applications of integration in cartesian and polar coordinates; curvature; differential equations of the first order and linear differential equations with constant coefficients; spherical trigonometry.

The text, as one may expect in a book of this kind, is extremely clearly written and no problem is left unsolved which will enable a future candidate to solve his examination difficulties. Little attempt is made to underline the mathematical difficulties inherent in the treatment here given but nevertheless there is a width of mathematical applications at all stages of the book which is refreshing to find in textbooks of this order. The examples are well graded and are spread over each chapter but an improvement would be effected if occasionally some miscellaneous problems were inserted so that a student would not automatically know which bag of tricks to use to solve the problem before him, particularly with reference to the extensive section on coordinate geometry. The examples are selected from a number of Universities.

The volume is well printed and attractively bound. Answers and hints are provided and an index covers the topics very comprehensively. Hints for background reading would help a reader who wished to do more than cram for an examination but in most cases perhaps a teacher might help in this matter.

The book would do well for those students who wish to study mathematics on traditional lines at a training college, for third year sixth form work and for candidates who require an introduction to the mathematics of the Mathematical Association's Diploma. C.H.

**A New Certificate Algebra. Vol. I**

Hislop. Methuen. With Answers 9/-. Without Answers 7/6.

This book is designed to cover the first half of an Algebra course leading to the Ordinary Level examination. The author states in his preface "——that a general picture is given of the whole range of the subject from Generalised Arithmetic to Quadratic Functions, though topics like factors, simplification of fractions, change of subject of formulae, the solution of quadratic equations, etc. are taken a step further in a subsequent volume."

The book has been well produced; the type is clear and the general lay-out is good. The introductory remarks on each topic have obviously been written with the pupil in mind, which does not always seem to be the case with books of this kind, and are written in a sufficiently informal manner for the reader to be able to use the text profitably for revision purposes.

*A NEW* Certificate Algebra does not imply, however, a new approach to the teaching of the subject, but rather an addition to the existing set of books which direct our pupils through the first two or three years. One chapter is an exception —the one concerned with graphs. Here, the author has successfully created an atmosphere where creative thinking may occur. But, even here, direction takes over when graphs of linear functions are considered, and opportunities of developing

mathematical awareness, which the situation offers, give way to the routine task of drawing graphs.

This book provides a sound foundation for teachers who are convinced that genuine algebraic thinking is stimulated by an approach which begins with Generalised Arithmetic and which concentrates on rules of procedure and on manipulation of algebraic notation before anything like an adequate reason is given for it. The examples for a course of this kind are well graded and there is a wide variety of problems which should satisfy the demands of any teacher.

But the time has come for EVERY teacher of mathematics to question this mechanical approach to the teaching of his subject. Much of what our pupils are compelled to do is not mathematics as such at all, but a passive, step by step acceptance of rules of procedure. By the time most of our pupils have left school, they have developed a lifeless, stunted way of thinking for which we must accept responsibility. How many children, with minds longing for adventure have taken a quick look at what passes for mathematics and, finding blind routine, have turned their attentions elsewhere? And what about us? As members of this association, we must ask ourselves the question, "Was it perhaps the love of routine and a satisfaction gained from uncreative thinking which made us specialise in mathematics? And if so, is it possible that we are passing on this mode of thought to our pupils?"

B.B.

**Elementary Algebra for College Students.**

I. Drooyan & W. Wooton. John Wiley & Sons, 1961.
pp. 272 + x 40s.

American College students, let it be hastily explained. The standard, in fact, is approximately to 'O' level of G.C.E. The book starts with chapters on natural numbers and their literal representation, and signed numbers, and proceeds through first degree equations, products and factors, fractions, simultaneous linear equations, quadratic equations and irrational numbers, to a final chapter on number systems.

The book is thorough, methods are explained, rules given, and illustrative examples interspersed with a good selection of examples for the student to work out. But perhaps it is too thorough: the student is required to do little thinking for himself —it is all there, he has only to digest it. The treatment is not particularly imaginative; for example, graphical methods are given simply as one method of solving equations. But there are good features, such as the insistence on the acquisition of a good and precise mathematical vocabulary, and the "cumulative review" examples at the end of each chapter which give progressive revision.

The book would be useful for a student who has to work alone with little or no tutorial assistance, but in spite of the high quality of its printing and binding, it is doubtful if it will find a ready sale in British classrooms at this high price.

C.B.

**Elements of Calculus**

T. S. Peterson. Harper & Bros.* 2nd edition, 1960. pp. 519 + xii. 46s.

**Calculus with Analytic Geometry**

T. S. Peterson. Harper & Bros. 1960* pp. 586 + xii 52s 6d.

* available in London through Hamish Hamilton.

These are two splendid books, well written and finely printed and bound. They are identical in text except that the larger one contains two extra chapters on analytic geometry — "Co-ordinates and Lines" and "Equations of the Second Degree." However, both volumes contain a chapter on solid analytic geometry, an essential feature since a later chapter deals with the applications of calculus to three-dimensional space. Although the calculus book stands well on its own, comparison of the two volumes indicates that the additional two chapters considerably enhance the value of the book. This is especially true of the chapter on second degree equations which gives a concise treatment of the conic sections. However, students working to this standard in this country would require to do more work on two dimensional geometry than there is given here. As far as the calculus goes this is a book for the First Year University student and for the mathematics specialist in the Sixth Form.

The book starts from first principles with a consideration of variables, functions and limits and covers the usual topics of differentiation and integration and their applications. There is full coverage and thorough treatment, and while it is impossible to list all the contents here, particular mention may be made of some of the more interesting topics. Thus we find integration introduced early in the course for the benefit of engineers and others who need the applications of integration for their other work. There is a full chapter on polar co-ordinates and applications to velocity and accelerations, a chapter on indeterminate forms (including Rolle's and Cauchy's Theorems), and a useful one on curve tracing. A chapter on Infinite Series gives the various convergency tests, and is followed by a consideration of expansions involving Taylor and Maclaurin's Theorems. There are also chapters on solid analytic geometry, partial differentiation, multiple integrals, differential equations, and a final useful chapter on vector analysis. The latter includes differentiation and integration of vectors, directional derivatives, line integrals and surface integrals. The books are prefaced with lists of mathematical formulae, including shapes of typical curves of equations, and end with a table of integrals, numerical tables and a very full index. Answers are given to all odd-numbered problems in the text; a pamphlet containing answers to the even-numbered ones is available.

We did not see these books in their first edition (1950) but it is clear that the author has used the preparation of the second edition as an opportunity for bringing the books up to date in mathematical technique. A pleasing feature is the way in which practical applications are brought into the work at all levels. Well-drawn illustrations are liberally used to bring out the essential principles, and the textual treatment is clear, concise and thorough. Every principle is fully illustrated by worked examples, while the examples for the student show variety and a desire to make the student think about the principles he has learnt.

The American origin of the book does not intrude itself unduly on the text, although the occasional phrase may cause the British student to stop to think (e.g. "The formulas $D_{10}$ through $D_{21}$ are . . . ").

These books have a place among the better calculus texts at this level.

C.B.

## The Philosophy of mathematics

S. Korner. Hutchinson 12/6.

The first task of a teacher of philosophy is very often to create in the mind of his student some awareness of a question. The student goes on to inspect various historical answers to the question and as their inadequacies are exposed his awareness of a problem increases. A contemporary solution sometimes dissipates the problem by finding the original question meaningless and recommending that it should not be asked. If this seems unfair on the student who never asked it in the first place it may be argued that such a training is as good as any other for the future district commissioner, politician or, for that matter, mathematician.

Few men are philosophers in the technical sense and the philosophy of mathematics is very technical. In general, working mathematicians are too busy asking mathematical questions to ask metamathematical ones (i.e. ones *about* mathematics). But in a teacher — at any level — a disinclination to stand back and comment about mathematics is fatal; it leads to formal teaching of the worst type. A study of the philosophy of mathematics can provide insight into the nature of mathematics. This insight may be reflected in one's teaching though such a study may mean learning not to ask questions one doesn't ask anyway.

The three current, strongly conflicting, views on the nature of mathematics remain apparently irreconcileable. But each illuminates mathematics in the classroom in its own way. Brouwer's intuitionist view of mathematics is revealed in our avoidance of a reductio ad absurdum proof whenever possible and his claim that large parts of classical analysis are invalid may be compared with the preference of a computor age for algorithms. Hilbert re-orientated geometrical teaching. Formalism remains a useful pragmatic philosophy in the sixth form; Kronecker would have delighted in many sections of the Mathematical Association reports. The Frege-Russell view of mathematics ("pure mathematics is the class of all propositions of the form $p$ implies $q$") is implied in most text-books of analysis today though some English ones incorporate a neoplatonic line of their own.

Prof. Korner's book will probably not find a place on the shelves of the school library but his careful exposition and criticism of each view will be most valuable to the interested teacher, who will note the reasonable price with gratitude. The first chapter gives a historical introduction for the philosophy student for whom the book is primarily intended. Mathematicians new to the subject may prefer to be introduced to the problem in a more technical way. Waismann's "Introduction to mathematical thinking" is a classic in this field.

In his introduction, Prof. Korner suggests that all philosophies of mathematics are developed, explicitly or implicitly, within the frame work of some wider philosophic system. A naive answer to the question "what is mathematics?" is that it is what mathematicians do and, as mathematicians, we may feel that this, or Wittgenstein's subtler version of it, will do. As philosophers, in a wider context, we may still be unsatisfied and Prof. Korner outlines a description of mathematics in terms of exact and inexact concepts and their relation to perception. These technical terms are defined and analysed in the last chapter.

"Mathematicians are engaged in filling two kinds of gap, not always sharply distinguishable — gaps consisting in the absence of theorems within existing theories, and gaps consisting in the absence of theories. Philosophical considerations are more likely to be influential when the task is not so much to find theorems, as to find theories." This is certainly the task today and this book will serve a useful purpose if it helps to stimulate mathematicians to think philosophically. It *would* be nice to put the choice axiom in its place.

D.T.

## Applied Mathematics

Hodge & Wood. Blackie, 1961. pp. 298+ix, 21s.

This book is well-written, well-produced and covers the familiar ground to A-level adequately. But what familiar ground it is! We have the same topics treated in the same way as we find in so many other applied mathematics books. Applied mathematics in our secondary schools seems to have got into a rut. Probably no other subject fits so tightly into the pattern set by examination syllabuses and questions.

Not that Messrs. Hodge and Wood are to blame; they appear to cater well for what is generally expected. In fact, they realise the shortcomings of the subject; for example, they start one chapter (Projectiles) by stating that the results therein have no practical application!

But the time has come to question what we are doing and ask where we are going. Much school applied mathematics has become mere technique: deciding which principle to use when. The *real* present-day applications of mathematics are to be found in such topics as statistics, linear programming, and the application of differential equations to physical and chemical problems. Thus in the present book, one of the more interesting chapters is where the authors apply differential equations to motion of a particle in a resisting medium, speed of chemical reactions, Newton's law of cooling and so on. A Sixth-former can find more interest and meaning in this than, for example, in mere composition and resolution of forces. An investigation of the place and content of applied mathematics in secondary schools is overdue.

Although mainly along traditional lines, this book is worthy of examination by those teachers taking this subject. In clarity and conciseness it has much to recommend it.

C.B.

## A.T.A.M. DIARY

1961

November 4th—*London Branch* at Avery Hill Training College, Horseferry Road, S.W.1. (Details — D. Fielker, 90 Burnet Ash Hill, S.E.12).

November 4th—*N.W. Group* at Manchester School of Education. (Details — Miss Clutten, 14 Bristol Street, Burnley).

November 25th—*Southampton* Institute of Education: Exhibition of Aids and Books. (Details — D. T. Moore).

November 25th—*Middlesex Branch* at Aylestone School, N.W.6. (Details — Miss Blandino, 9 Barnhill Road, Wembley Park).

November 29th—*Maidstone* — Boys' Technical School. (Details — D. T. Moore).

1962

January 27th—*London* (Provisional date).

April 16—19th—*A.G.M. and Easter Conference* in Manchester.

*Conference Secretary:* D. T. Moore, 234 Birling Road, Snodland, Kent.

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## ANNOUNCEMENTS

A new branch of the Association has been formed in West Sussex and the inaugural conference is to be held at the Blessed Philip Howard School, Barnham, on Saturday, 11th November. Applications for membership of the branch should be sent to Mr. N. D. Anderson, Bognor Regis Grammar School, or further details may be had from Mr. J. L. Dewar, William Fletcher School, Bognor Regis.

## SCHOOLS TV

The Editor would like to hear from teachers in any schools taking the BBC tv series on *Mathematics and Life* which is being transmitted during the Autumn Term.

## A.G.M. AND EASTER CONFERENCE

The Annual General Meeting and Easter Conference of the Association will be held in Manchester from April 16th to 19th, 1962. Full details will be given in our next issue.

The Association for Teaching Aids in Mathematics is always interested in arranging meetings for teachers of mathematics in any part of the country. There is a panel of speakers willing to talk on a wide variety of topics relating to the teaching of the subject and also give demonstration lessons. The Association co-operates with Local Education Authorities, Training Colleges, University Education Departments, etc., and any such or any individuals who feel there is need for a meeting in a particular area are asked to get in touch with the Association's Conference Secretary, Mr. D. T. Moore, 234 Birling Road, The Groves, Snodland, Kent.

## Advanced Pure Mathematics

*J. D. HODSON, M.A., B.Sc.*

This book is a direct successor to the author's *Elementary Pure Mathematics*. It completes the course up to Scholarship standard. 124 *Figures*, **30***s*.

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## An Introduction to Statistical Methods

*H. J. HALSTEAD, B.A., Dip.Ed. (Melb.)*

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**MACMILLAN & CO. LTD.** *ST. MARTIN'S STREET, LONDON, W.C.2*

## *Reviewer after Reviewer*

has praised THE LANGUAGE OF MATHEMATICS by Dr F. W. Land.

Here are extracts from two recent reviews.

'There is no maths teacher, in school or technical college, whose capabilities would not be enhanced by the stimulation of reading it. For young craftsmen and technicians who, more than most, get their maths in a utilitarian reach-me-down mode it could be a challenging and enlivening experience. It would indeed make an altogether worthy apprenticeship prize, especially as the design, typography and illustration are quite outstanding.' *Technology, June* 1961.

'It is in every respect a delightful book; merely to turn its pages is an attractive occupation, and the reader is drawn again and again to browse, to ponder and then to read with understanding. Little in the way of prior knowledge or acquired ability in mathematics is necessary from those who begin. If they are prepared to think as they read, the author's enthusiasm and clarity of expression, supported by Margaret Clark's beautiful illustrations, will carry them along. I read it at a single sitting and have no hesitation in saying that it should form part of the general studies course in every sixth form, as well as being a source of pleasure and profit to every intelligent adult.'

*Nature, August* 1961.

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